The Structure of Certain Components of Springer Fibers

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(Joint work with L. Barchini and with W. Graham)

I.

- G a complex reductive algebraic group.
- $\mathfrak{g}$ the Lie algebra of G.
- $\mathfrak{B}$ the (full) flag variety of G.
- $T^*\mathfrak{B} = \{(\mathfrak{b}, \xi) : \xi \in (\mathfrak{g}/\mathfrak{b})^*\}$ the cotangent bundle.

The moment map for the action of G on $T^*\mathfrak{B}$ is

$$\mu : T^*\mathfrak{B} \to \mathfrak{g}^*$$
$$\mu(\mathfrak{b}, \xi) = \xi.$$

Identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ via a nondegenerate G-invariant form gives

$$T^*\mathfrak{B} = \{(\mathfrak{b}, \xi) : \xi \in \mathfrak{n}^-, \mathfrak{n}^- \text{ the nilradical of } \mathfrak{b}\}$$
$$\mu(\mathfrak{b}, Y) = Y \in \mathcal{N} = \text{ (nilpotent cone in } \mathfrak{g}).$$

A few well-known facts:

(1) μ a proper birational map (Springer, 1969).

(2) The fibers are 'equidimensional', i.e., the irreducible components of a given fiber all have the same dimension (Spaltenstein, 1977).

(3) The fibers are connected projective varieties.

(4) The components need not be smooth.

Note that the fiber $\mu^{-1}(Y)$ is the set of Borel subalgebras containing Y.

A (slightly) different description of μ may be given by writing the cotangent bundle as an induced bundle: fix a Borel subalgebra $\mathfrak{b}$,

$$T^*\mathfrak{B} \simeq G \underset{B}{\times} \mathfrak{n}^-$$

then

$$\mu(g, Y) = \mathrm{Ad}(g)Y \in \mathcal{N}.$$

In this case, defining

$$N(Y, \mathfrak{n}^-) = \{g \in G : \mathrm{Ad}(g)Y \in \mathfrak{n}^-\},$$

we have

$$\mu^{-1}(Y) = (N(Y, \mathfrak{n}^-))^{-1} \cdot \mathfrak{b}$$

$G_{\mathbf{R}}$ a real reductive linear group,
$K_{\mathbf{R}}$ a maximal compact subgroup.
G the complexification and
θ the complexification of the Cartan involution.
$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan decompositon
$K = G^{\theta}$.
$\mathcal{N}_{\theta} = \mathcal{N} \cap \mathfrak{p}$.

It is natural to consider K-orbits $\mathcal{Q}$ in $\mathfrak{B}$ and the corresponding conormal bundles. Write $\mathcal{Q} = K \cdot \mathfrak{b}$, then

$$T^*_{\mathcal{Q}}\mathfrak{B} \simeq K \underset{K \cap B}{\times} (\mathfrak{n}^- \cap \mathfrak{p}).$$

Then for $Y \in \mathcal{N}_{\theta}$,

$$\mu^{-1}(Y) = \bigcup_{\mathcal{Q}} \left(\mu^{-1}(Y) \cap \overline{T^*_{\mathcal{Q}}\mathfrak{B}}\right).$$

Each $\mu^{-1}(Y) \cap \overline{T^*_{\mathcal{Q}}\mathfrak{B}}$ is a union of several components of $\mu^{-1}(Y)$.

We restrict our attention to **closed** K-orbits in $\mathfrak{B}$. The corresponding components play an important role in the computation of associated cycles of discrete series representations.

So, let's fix a closed K-orbit $\mathcal{Q} = K \cdot \mathfrak{b}$ in $\mathfrak{B}$ and restrict the moment map to the conormal bundle:

$$\mu_{\mathcal{Q}} : T^*_{\mathcal{Q}}\mathfrak{B} \to \mathcal{N}_\theta$$
$$\mu(k, Y) = \mathrm{Ad}(k)Y \in \mathcal{N}_\theta.$$

Then $\mu_{\mathcal{Q}}^{-1}(Y) = (N(Y, \mathfrak{n}^- \cap \mathfrak{p}))^{-1} \cdot \mathfrak{b} \subset \mathcal{Q}$. Where $N(Y, \mathfrak{n}^- \cap \mathfrak{p}) = \{k \in K \ : \ \mathrm{Ad}(k)Y \in \mathfrak{n}^- \cap \mathfrak{p}\}$.

In this setting we define an element $f \in \mathfrak{n}^- \cap \mathfrak{p}$ to be *generic in* $\mathfrak{n}^- \cap \mathfrak{p}$ if the image of $\mu_{\mathcal{Q}}$ is $\overline{K \cdot f}$.

⊣⊢

Recall that closed orbits of K in a flag variety may be described by choosing a CSA $\mathfrak{h}$ in $\mathfrak{k}$ (let's assume this is a CSA of $\mathfrak{g}$ as well). Then fix a positive system Δ_c^+ of roots in $\Delta(\mathfrak{h}, \mathfrak{k})$. Then, the closed orbits are in one-to-one correspondence with the positive systems $\Delta^+ \subset \Delta(\mathfrak{h}, \mathfrak{g})$ that contain Δ_c^+.

$$\Delta^+ \rightsquigarrow K \cdot \mathfrak{b}, \ \ \mathfrak{b} = \mathfrak{h} + \mathfrak{n}^- = \mathfrak{h} + \sum_{\alpha \in \Delta^+} \mathfrak{g}^{(-\alpha)}.$$

Such positive systems are in one-to-one correspondence with the Δ_c^+-dominant W-conjugates of ρ.

II. Description of Fiber.

For the rest of the lecture we consider the pair

$$(G, K) = (GL(n),\, GL(p) \times GL(q))$$

(coming form $G_{\mathbf{R}} = U(p, q)$).

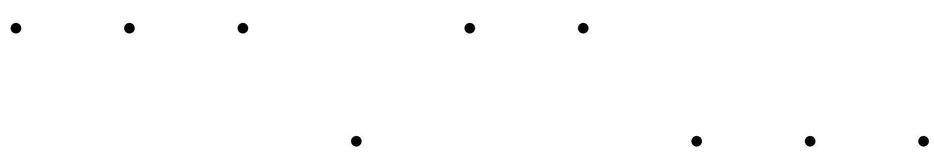
The closed K-orbits correspond to W-conjugates λ of $(n, n-1, \ldots, 2, 1)$:

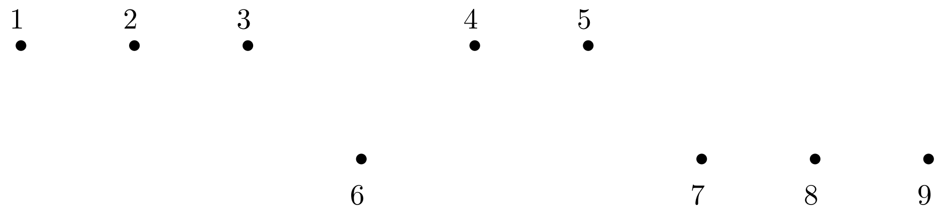
$$\lambda = (a_1, \ldots, a_p, b_1, \ldots, b_q)$$

with $a_1 > \cdots > a_p$, $b_1 > \cdots > b_q$. This λ determines a positive system of roots and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^-$ and $\mathcal{Q} = K \cdot \mathfrak{b}$.

- Find an f generic in $\mathfrak{n}^- \cap \mathfrak{p}$.
- Describe the fiber $\mu_{\mathcal{Q}}^{-1}(f)$.

Here is an example. Consider $(GL(9)\,, GL(5)\times GL(4))$.
Let $\lambda = (9, 8, 7, 5, 4 \,|\, 6, 3, 2, 1)$.
All the information is in the following array





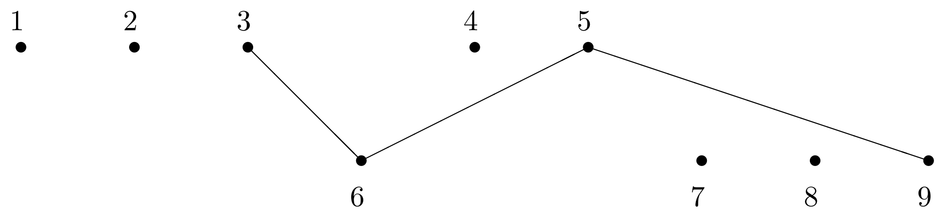
The flag stabilized by $\mathfrak{b}$ is read off of this array.

Choose a 'string' through the diagram by choosing the farthest right dot in each 'block'. Write the labels of the corresponding dots as $k_1, k_2, \ldots, k_\ell$ (left to right). Then

$$f_0 = \sum_{j=1}^{\ell-1} E_{k_{j+1},k_j},$$

with $E_{i,j}$ the standard root vector for the root $\epsilon_i - \epsilon_j$.

In the example this looks like



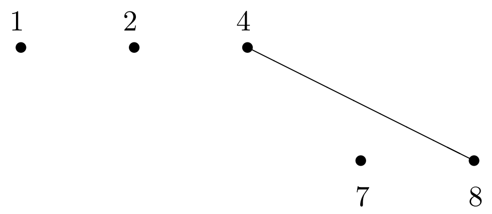
$$f_0 = E_{9,5} + E_{5,6} + E_{6,3}.$$

- Form a parabolic subgroup $Q = LU^-$ of K by specifying $\Delta(\mathfrak{l})$ is spanned by the roots in $\mathfrak{k}$ that are simple roots for $\Delta^+(\lambda)$.

$$L \simeq GL(3) \times GL(2) \times GL(1) \times GL(3).$$

- Form G_1, a smaller $GL(n_1)$ commuting with f_0.
- Choose f_1 inside $\mathfrak{g}_1$ by the same procedure.

Now delete from the array the dots already hit, and do it again:



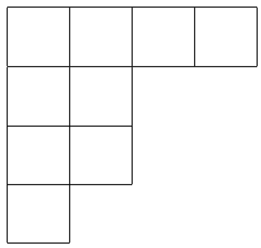
$f_1 = E_{8-4}$

And again to get $f_2 = E_{7,2}$.

Then $f = f_0 + f_1 + f_2$, and the action of f can be expressed as

$$\begin{aligned} f : e_3 &\to e_6 \to e_5 \to e_9 \to 0 \\ e_4 &\to e_8 \to 0 \\ e_2 &\to e_7 \to 0 \\ e_1 &\to 0. \end{aligned}$$

The tableau associated to f (row lengths are the sizes of Jordan blocks) is



Inside G_1 construct $Q_1 = L_1 U_1^-$, as Q was constructed in G....$\Delta(\mathfrak{l}_1)$ is spanned by the compact roots simple in $\Delta(\mathfrak{g}_1)$.

Note that $L_1 \simeq GL(3) \times GL(2)$, and

$$L_1 \text{ is NOT a subgroup of } L.$$

Theorem 1 (Barchini-Z.) *This f is generic in $\mathfrak{n}^- \cap \mathfrak{p}$.*

Furthermore, if the parabolic subgroups $Q_i = L_i U_i^-$ are formed working in G_i (and we set $Q_0 = Q, L_0 = L$) then

Theorem 2 (Barchini, Z.)

$$\mu_{\mathcal{Q}}^{-1}(f) = Q_m \cdots Q_1 Q_0 \cdot \mathfrak{b} = L_m \cdots L_1 L \cdot \mathfrak{b}.$$

Sketch of Proof.

- $\mu_{\mathcal{Q}}^{-1}(f)$ is irreducible by some general argument and the fact that the component group of the centralizer of f in K is trivial.
- $L_m \cdots L_1 L \cdot \mathfrak{b} \subset \mu_{\mathcal{Q}}^{-1}(f)$, follows by detailed understanding of the construction. Also, $L_m \cdots L_1 L \cdot \mathfrak{b} = Q_m \cdots Q_1 Q \cdot \mathfrak{b}$.
- $L_m \cdots L_1 L \cdot \mathfrak{b}$ and $\mu_{\mathcal{Q}}^{-1}(f)$ have the same dimension.
- $L_m \cdots L_1 L \cdot \mathfrak{b}$ is closed.

The last two use a simple fact: $R_i \equiv L_i \cap Q_{i-1}$ is a parabolic subgroup of L_i (define $R_0 = L \cap B$).

Write $\mathfrak{v}_i$ for the nilradical of the opposite parabolic. Then we have the formula

$$\dim(\mu_{\mathcal{Q}}^{-1}(f)) = \sum_{j=0}^{m} \dim(L_j/R_j)$$

and $L_m \cdots L_1 L \cdot \mathfrak{b}$ contains the subset

$$\exp(\mathfrak{v}_m) \cdots \exp(\mathfrak{v}_1) \exp(\mathfrak{v}_0) \cdot \mathfrak{b},$$

having the same dimension. □

Remark. The nilradical can easily be described in terms of the array.

III.

We can say more (joint work with W. Graham).

Define $R_i' = Q_i \cap Q_{i-1}$, and $R_0' = Q_0 \cap B$.
Consider $Q_m \times \cdots Q_1 \times Q_0$ ($Q_0 = Q$), with the action of $R_m' \times \cdots R_1' \times R_0'$ given by

$$(r_m, r_{m-1}, \ldots, r_1, r_0) \cdot (q_m, q_{m-1}, \ldots, q_1, q_0) \\ = (q_m r_m^{-1}, r_m q_{m-1} r_{m-1}^{-1}, \ldots, r_2 q_1 r_1^{-1}, r_1 q_0 r_0^{-1}).$$

We denote the quotient by

$$X = Q_m \underset{R_m}{\times} Q_{m-1} \underset{R_{m-1}}{\times} \cdots \underset{R_2'}{\times} Q_1 \underset{R_1'}{\times} Q_0/R_0'.$$

X is a smooth projective variety by some generalities.

(Note: $Q_i/R_i' \simeq L_i/R_i$ is a (gen) flag variety for L_i.)

The map $(q_m, \ldots, q_1, q_0) \mapsto q_m \ldots q_1 q_0 \cdot \mathfrak{b}$ is a surjection from $Q_m \times \cdots \times Q_1 \times Q_0$ onto $\mu_{\mathcal{Q}}^{-1}(f)$ and clearly gives a surjection

$$F : X \to \mu_{\mathcal{Q}}^{-1}(f).$$

Theorem 3 *The map $F : X \to \mu_{\mathcal{Q}}^{-1}(f)$ is an isomorphism of varieties.*

Sketch of Proof. That F is a bijection follows from a detailed understanding of the construction of f (and the L_i's,...).

To see that F is an algebraic isomorphism, we apply the following fact from algebraic geometry:

> Suppose Z is projective and $\varphi : Z \to W$ is a morphism that is bijective. If $\varphi_* \mathcal{O}_Z = \mathcal{O}_W$, then φ is an isomorphism.

For this it suffices to show

$$H^0(X, F^*(\mathcal{O}(\mathcal{L}_\tau))) \simeq H^0(\mu_{\mathcal{Q}}^{-1}(f), \mathcal{O}(\mathcal{L}_\tau))),$$

for $0 \ll \tau$. The L.H.S is computed by Borel-Weil applied to the iterated bundle and the R.H.S. is computed in B-Z, 2008. □

Corollary 4 $\mu_{\mathcal{Q}}^{-1}(f)$ *is a smooth variety.*

IV. Action of a Torus.

Fact: The maximal torus H (= diagonals) does not act on all components of Springer fibers.

Proposition 5 (Graham-Z.) *If $\mathcal{Q}$ is a closed K-orbit in $\mathfrak{B}$, then H acts on $\mu_{\mathcal{Q}}^{-1}(f)$. The fixed point set of this action is*

$$\{w_m \dots w_1 w_0 \cdot \mathfrak{b} : w_i \in W(L_i)\}.$$

There are formulas for the action of H on tangent spaces at the fixed points.

IV. Standard Tableaux.

For $GL(n)$ the components of the Springer fibers are parameterized by 'standard' tableaux (Steinberg, Spaltenstein). This is done as follows. Fix a nilpotent element f.

(i) $\mu^{-1}(f)$ is the set of $\mathfrak{b}$ containing f.

(ii) Each $\mathfrak{b}$ corresponds to a flag of subspaces in $\mathbf{C}^n$;

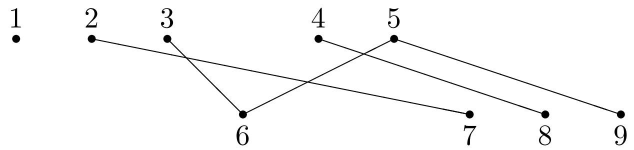
$$\{0\} \subset F_1 \subset \cdots \subset F_n = \mathbf{C}^n.$$

(iii) The tableau (without numbers) of each $f|_{F_i}$ is obtained from that of $f|_{F_{i-1}}$ by adding a block (onto the end of a row or so as to begin a new row).

(v) Begin with $f|_{F_1}$, the tableau is one block, place '1' in this block. Then place an 'i' in the new block in passing from $f|_{F_{i-1}}$ to $f|_{F_i}$.

We'll refer to this standard tableau associated to $\mathfrak{b} \in \mu^{-1}(f)$ as $ST(\mathfrak{b}, f)$

In the earlier example the array is



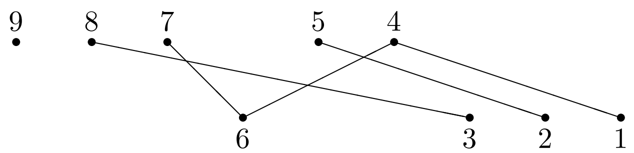
And recall that

$$\begin{aligned} f : e_3 &\rightarrow e_6 \rightarrow e_5 \rightarrow e_9 \rightarrow 0 \\ e_4 &\rightarrow e_8 \rightarrow 0 \\ e_2 &\rightarrow e_7 \rightarrow 0 \\ e_1 &\rightarrow 0. \end{aligned}$$

Then the standard tableau for the base point $\mathfrak{b}$ of $\mathcal{Q}$ is $ST(\mathfrak{b}, f)$:

1	4	6	7
2	5		
3	8		
9			

This is read off the array by renumbering:



Proposition 6 *The above algorithm gives the standard tableau associated to the component $\mu_{\mathcal{Q}}^{-1}(f)$.*

Proof. Need to find an open subset of $\mathfrak{b}'$ in $\mu_{\mathcal{Q}}^{-1}(f)$ on which the standard tableau $ST(\mathfrak{b}', f)$ is the one specified. This open set is

$$\exp(\mathfrak{v}_m) \cdots \exp(\mathfrak{v}_1) \exp(\mathfrak{v}_0) \cdot \mathfrak{b}$$

□

V. An Example.

The parabolic subgroup Q (of K, $\Delta(\mathfrak{l})$ spanned by the compact roots that are simple for Δ^+) normalizes $\mathfrak{n}^- \cap \mathfrak{p}$. Therefore, $Q \cdot f$ is some subset of $\mathfrak{n}^- \cap \mathfrak{p}$.

It is natural to ask:

- Does Q have a dense orbit in $\mathfrak{n}^- \cap \mathfrak{p}$?
- Is $Q \cdot f$ is equal to all of the generic elements in $\mathfrak{n}^- \cap \mathfrak{p}$?

An example of Tauvel, 2001, shows that a Borel does not have a dense orbit.

Proposition 7 (Barchini-Z.) *Q has an open orbit in $\mathfrak{n}^- \cap \mathfrak{p}$ if and only if $Q_1 \cap Q$ has an open orbit in $\mathfrak{n}_1 \cap \mathfrak{p}$.*

There is an example in $(GL(14), GL(7) \times GL(7))$ for which Q has no dense orbit in $\mathfrak{n}^- \cap \mathfrak{p}$. Take $\lambda = (14, 13, 10, 7, 6, 4, 3 \,|\, 12, 11, 9, 8, 5, 2, 1)$. This is the smallest example.