DISCRETE COMPONENTS OF SOME COMPLEMENTARY SERIES REPRESENTATIONS

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Abstract. We show that the restriction of the complementary reries representations of $SO(n,1)$ to $SO(m,1)$ $(m < n)$ contains complementary series representations of $SO(m,1)$ discretely, provided that the continuous parameter is sufficiently close to the first point of reducibility and the representation of M- the compact part of the Levi- is a sufficiently small fundamental representation.

We prove, as a consequence, that the cohomological representation of degree i of the group $SO(n,1)$ contains discretely, for $i \leq m/2$, the cohomological representation of degree i of the subgroup $SO(m,1)$ if $i \leq m/2$.

As a global application, we show that if $G/\mathbb{Q}$ is a semisimple algebraic group such that $G(\mathbb{R}) = SO(n,1)$ up to compact factors, and if we assume that for all n, the tempered cohomological representations are not limits of complementary series *in the automorphic dual of* $SO(n,1)$, then for all n, non-tempered cohomological representations are isolated in the automorphic dual of G. This reduces conjectures of Bergeron to the case of **tempered** cohomological representations.

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1. Introduction

A famous Theorem of Selberg [Sel] says that the non-zero eigenvalues of the Laplacian acting on functions on quotients of the upper half plane $\mathfrak{h}$ by *congruence subgroups* of the integral modular group, are bounded away from zero, as the congruence subgroup varies. In fact, Selberg proved that every non-zero eigenvalue λ of the Laplacian on functions on $\Gamma\backslash\mathfrak{h}$, $\Gamma \subset SL_2(\mathbb{Z})$, satisfies the inequality

$$\lambda \geq \frac{3}{16}.$$

A Theorem of Clozel [Clo] generalises this result to any congreunce quotient of any symmetric space of non-compact type: if G is a linear semi-simple group defined over $\mathbb{Q}$, $\Gamma \subset G(\mathbb{Z})$ a congruence subgroup and $X = G(\mathbb{R})/K$ the symmetric space of G, then nonzero eigenvalues λ of the Laplacian *acting on the space of functions on* $\Gamma\backslash X$ satisfy:

$$\lambda \geq \epsilon,$$

where $\epsilon > 0$ is a number **independent** of the congruence subgroup Γ.

Analogous questions on Laplacians acting on differential forms of higher degree (functions may be thought of as differential forms of degree zero) have geometric implications for the cohomology of the locally symmetric space. Concerning the eigenvalues of the Laplacian, Bergeron ([Ber]) has made the following conjecture:

Conjecture 1. (Bergeron) Let X be the real hyperbolic n-space and $\Gamma \subset SO(n,1)$ a congruence arithmetic subgroup. Then non-zero eigenvalues λ of the Laplacian acting on the space $\Omega^i(\Gamma\backslash X)$ of differential forms of degree i satisfy:

$$\lambda \geq \epsilon,$$

for some $\epsilon > 0$ **independent** of the congruence subgroup Γ, provided i is strictly less than the "middle dimension (i.e. $i < [n/2]$).

If n is even, the above conclusion is conjectured to hold even if $i = [n/2] = n/2$. (When n is odd, there is a slightly more technical statement which we omit).

In this paper, we show, for example, that if the above conjecture holds true in the *middle degree* for all even integers n, then the conjecture holds for differential forms of arbitrary degrees (See Theorem 3). For odd n, there is, again, a more technical statement (see Theorem 3).

The main Theorem of the present paper is Theorem 1 on the occurrence of discrete components in the restriction of of certain complementary series representations of $SO(n,1)$ to $SO(m,1)$. The statement on Laplacians may be deduced from the main theorem, using the Burger-Sarnak method ([Bu-Sa]).

We now describe the main theorem more precisely. Let $i \leq [n/2]-1$ and $G = SO(n,1)$. Let $P = MAN$ be the Langlands decomposition of G, $K \subset G$ a maximal compact subgroup of G containing M. Let $\mathfrak{p}_M$ be the standard representation of M and $\wedge^i$ be its i-th exterior power representation. Denote by ρ_P^2 the character of P acting on the top exterior power of the Lie algebra of N. Consider the representation

$$\widehat{\pi_u(i)} = Ind_P^G(\wedge^i \otimes \rho_P(a)^u)$$

for $0 < u < 1 - \frac{2i}{n-1}$. The representation $\widehat{\pi_u(i)}$ denotes the **completion** of the space π_u of K-finite vectors with respect to the G-invariant metric on $\pi_u(i)$, and is called the **complementary series representation** corresponding to the representation $\wedge^i$ of M and the parameter u.

Let $H = SO(n-1,1)$ be embedded in G such that $P \cap H$ is a maximal parabolic subgroup of H, $A \subset H$, and $K \cap H$ a maximal compact subgroup of H. We now assume that $\frac{1}{n-1} < u < 1 - \frac{2i}{n-1}$. Let $u' = \frac{(n-1)u-1}{n-2}$; then $0 < u' < 1 - \frac{2i}{n-2}$. Denote by $\wedge_H^i$ the i-th exterior power of the standard representation of $M \cap H \simeq O(n-2)$.

We obtain analogously the complementary series representation

$$\widehat{\sigma_{u'}(i)} = Ind_{P\cap H}^H(\wedge_H^i \otimes \rho_{P\cap H}(a)^{u'}),$$

of H. The main theorem of the paper is the following.

Theorem 1. *If*

$$\frac{1}{n-1} < u < 1 - \frac{2i}{n-1},$$

then the complementary series representation $\widehat{\sigma_{u'}(i)}$ occurs discretely in the restriction of the complementary series representation $\widehat{\pi_u(i)}$ of $G = SO(n,1)$ to the subgroup $H = SO(n-1,1)$:

$$\widehat{\sigma_{u'}(i)} \subset \widehat{\pi_u(i)}_{|SO(n-1,1)}.$$

Remark. The corresponding statement is false for the space of K-finite vectors in both the spaces; this inclusion holds only at the level of completions of the representations involved.

Denote by $A_j(n)$ the unique unitary cohomological representation of G which has cohomology (with trivial coefficients) in degree j. As u tends to the limit $1-\frac{2i}{n-1}$, the representations $\pi_u(i)$ tend (in the Fell topology on the Unitary dual $\widehat{G}$) both to the representation $A_i = A_i(n)$ and to $A_{i+1} = A_{i+1}(n)$. Using this, and the proof of Theorem 1, we obtain

Theorem 2. *The restriction of the cohomological representation $A_i(n)$ of $SO(n,1)$ to the subgroup $H = SO(n-1,1)$ contains discretely, the cohomological representation $A_i(n-1)$ of $SO(n-1,1)$:*

$$A_i(n-1) \subset A_i(n)_{|SO(n-1,1)}.$$

Supose now that G is a semi-simple linear algebraic group defined over $\mathbb{Q}$ and $\mathbb{Q}$-simple, such that

$$G(\mathbb{R}) \simeq SO(n,1)$$

up to compact factors (if $n=7$, we assume in addition that G is not the inner form of some trialitarian D_4). Then there exists a $\mathbb{Q}$-simple $\mathbb{Q}$-subgroup $H_1 \subset G$ such that

$$H_1(\mathbb{R}) \simeq SO(n-2,1),$$

up to compact factors.

Denote by $\widehat{G}_{Aut}$ the "automorphic dual" of $SO(n,1)$, in the sense of Burger-Sarnak ([Bu-Sa]). Suppose $A_i = A_i(n)$ is a limit of representations ρ_m in $\widehat{G}_{Aut}$. The structure of the unitary dual of $SO(n,1)$ shows that this means that $\rho_m = \widehat{\pi_{u_m}(i)}$ for some sequence u_m which tends from the left, to $1-\frac{2i}{n-1}$ (or to $1-\frac{2i+2}{n-1}$; we will concentrate only on the first case, for ease of exposition).

Since $\rho_m = \widehat{\pi}_{u_m}(i) \in \widehat{G}_{Aut}$, a result of Burger-Sarnak ([Bu-Sa]) (in the reference [Bu-Sa], the restriction $\rho_m|_H$ refers to the closure of the union of all the irreducibles of H which occur weakly in ρ_m restricted to H, in the Fell topology on the unitary representations of H), implies that

$$\rho_m|_H \subset \widehat{H}_{Aut}.$$

Applying Theorem 1 twice, we get

$$\sigma_m = \widehat{\sigma}_{u''_m}(i) \in \rho_m|_H \subset \widehat{H}_{Aut}$$

Taking limits as m tends to infinity, we get $A_i(n-2)$ as a limit of representations σ_m in $\widehat{H}_{Aut}$. Therefore,the isolation of $A_i(n)$ in $\widehat{G}_{Aut}$ is reduced to that for $SO(n-2,1),\cdots$. We can finally assume that $A_i(m)$

is a tempered representation of $SO(m,1)$ where $m = 2i$ or $2i+1$. This proves the following Theorem.

Theorem 3. *If for all m, the* **tempered** *cohomological representation $A_i(m)$ (i.e. when $i = [m/2]$) is not a limit of complementary series in the automorphic dual of $SO(m,1)$, then for all integers n, and for $i < [n/2]$, the cohomological representation $A_i(n)$ is isolated in the automorphic dual of $SO(n,1)$.*

The proof of Theorem 1 is somewhat roundabout and proceeds as follows.

(1) We first prove Theorem 1 when $i = 0$; that is $\widehat{\pi}_u = \widehat{\pi}_u(i)$ is an unramified representation and $\frac{1}{n-1} < u < 1$. In this case, we get a model of the representation $\widehat{\pi}_u$ by restricting the functions (sections of a line bundle) on G/P to the big Bruhat cell $Nw \simeq \mathbb{R}^{n-1}$ and taking their Fourier transforms. The G- invariant metric is particularly easy to work with on the Fourier transforms; it is then easy to see that $\widehat{\sigma}_{u'}$ embeds isometrically in $\widehat{\pi}_u$.
(2) When this is interpreted in the space π_u of K-finite vectors, we have the isomorphism of the intertwining map $I_G(u)$:

$$I_G(u) : \pi_u \simeq \pi_{-u} = Ind_P^G(\rho_P^{-u}).$$

The restriction of the sections on G/P of the line bundle corresponding to π_{-u} to the subspace $H/H \cap P$ is exactly the representation $\sigma_{-u'}$, which is isomorphic, via the inverse of the intertwining map, namely $I_H(u')^{-1}$, to $\sigma_{u'}$. The existence of the embedding $\widehat{\sigma}_{u'} \subset \widehat{\pi}_u$ (and the multiplicity one for representations of $SO(n-1,1)$ occurring in $SO(n,1)$), imply that the restriction map of sections

$$res : \pi_{-u} \to \sigma_{-u'},$$

is continuous for the metric on $\pi_{-u} \simeq \pi_u$ and on $\sigma_{-u'} \simeq \sigma_{u'}$.

(2) The K-irreducible representations occurring in π_u are parametrised by non-negative integers m, each ireducible V_m occurring with multiplicity one (V_m is isomorphic to the space of homogeneous harmonic polynomials of degree m on the sphere $G/P \simeq K/M$): $\pi_u = \oplus_{m \geq 0} V_m$.

Similarly, we write $\sigma_{u'}$ as a direct sum of irreducibles W_l of $K \cap H$ (indexed again by non-negative integers): $\sigma_{u'} = \oplus_{l \geq 0} W_l$. Denote by $V_{m,l}$ the isotypical component of W_l in the restriction of V_m to $K \cap H$ (V_m restricted to $K \cap H$ is in fact multiplicity free). We have the restriction map $\widehat{r} : \pi = Ind_M^K(triv) \to \sigma = Ind_{M \cap H}^{K \cap H}(Triv)$. This maps

$V_{m,l}$ into W_l. Set

$$C(m,l,0) = \frac{|| \widehat{r}(f) ||^2_{K\cap H}}{|| f ||^2_K},$$

We show that the continuity of the map $res : \pi_{-u} \longrightarrow \sigma_{-u'}$ is equivalent to the statement that the series

$$\sum_{m\geq l} C(m,l,0)\frac{l^{(n-2)u'}}{m^{(n-1)u}} < A, \tag{1}$$

where A is a constant independent of the integer l.

The equivalence is proved by calculating the value of the intertwining operator on each K-type V_m (the operator acts by a scalar $\lambda_m(u)$ as a ratio of values of the classical Gamma function) and obtaining asymptotics as m tends to infinity by using Stirling's Approximation to the classical Gamma function.

Since, by the result quoted above, the restriction map is indeed continuous, the estimate of equation (1) is indeed true.

(3) We now describe the proof of Theorem 1 in the ramified case. We have analogously the restriction maps

$$r_u(i) : \pi_{-u}(i) \longrightarrow \sigma_{-u'}(i),$$

where again $u' = \frac{(n-1)u-1}{n-2}$. By the multiplicity one statement for irreducible representations of $SO(n-1,1)$ which are quotients of the representation $\pi_{-u}(i)$, it follows that Theorem 1 is equivalent to the continuity of the restriction map $r_u(i)$.

As a K-module, again $\pi_u(i)$ is a direct sum of irreducibles parametrised by non-negative integers and we can define the numbers $C(m,l,i)$ analogous to the definition of $C(m,l,0)$ above. We prove analogously that the continuity of the restriction $r_u(i)$ is equivalent to the estimate

$$\sum_{m\geq l} C(m,l,i)\frac{l^{(n-2)u'}}{m^{(n-1)u}} < A, \tag{2}$$

where A is a constant independent of the $K \cap H$ "type" l. The proof of this equivalence uses some precise estimates of the intertwining operators on the analogues of the spaces V_m in $\pi(i) \simeq Ind_M^K(\wedge^i)$ (and on the analogues of W_l in $\sigma(i)$ which is defined similarly to $\pi(i)$, but for the group H).

(4) We prove that the estimate in equation (2) is indeed true, by analysing the K-types occurring in $\pi(i)$, and showing as a consequence

that

$$C(m, l, i) \leq \gamma C(m, l, 0),$$

for some constant γ independent of m, l. Since equation 1) is proved to be true, it follows that equation (2) also is true. This proves Theorem 1 for arbitrary i.

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References

[Ber] N.Bergeron, Lefschetz properties for real and complex hyperbolic manifolds, Int. Math. Res. Not. (2003) no. 20., pp 1089-1122.

[BLS] Burger, M., Li, J-S. and Sarnak, P. Ramanujan Duals and automorphic spectrum, Bull.Amer.Math.Soc. (N.S.)**26** (1992), 253-257.

[Bu-Sa] Burger, M. and Sarnak, P. Ramanujan Duals II. Invvent.Math.**106** (1991) 1-11.

[Clo] Clozel,L. Démonstration de la conjecture τ. Invent. Math.**151** (2003), 297-328.

[GW] Goldman and Wallach, Representations and Invariants of Classical Groups, Encyclopedia of Mathematics and Its Applications, **68**, Cambridge University Press, Cambridge, 1998.

[Sel] A. Selberg, On the estimation of Fourier coefficients of modular forms, Proc. Sym. Pure Math. Vol VIII, Amer. Math. Soc., Providence, RI (1965), p 1-15.

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