Representation theoretic harmonic spinors for coherent families

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Abstract

Coherent continuation π_2 of a representation π_1 of a semisimple Lie algebra arises by tensoring π_1 with a finite dimensional representation F and projecting it to the eigenspace of a particular infinitesimal character. Some relations exist between the spaces of harmonic spinors (involving Kostant's cubic Dirac operator and the usual Dirac operator) with coefficients in the three modules. For the usual Dirac operator we illustrate with the example of cohomological representations by using their construction as generalized Enright-Varadarajan modules. In [9] we considered only discrete series, which arises as generalized Enright-Varadarajan modules in the particular case when the parabolic subalgebra is just a Borel subalgebra.

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1 Introduction.

Let G be a connected non-compact semisimple real Lie group with finite center and K a maximal compact subgroup. We assume G to be linear and also assume, though not essential, that both G and K have the same complex rank. Denote the Lie algebras by $\mathfrak{g}$ and $\mathfrak{k}$ respectively. We write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{p} = \mathfrak{k}^{\perp}$$

for the corresponding orthogonal decomposition (Cartan decomposition) of $\mathfrak{g}$ with respect to the Killing form. Let θ denote the Cartan involution. Note that the killing form is positive definite on $\mathfrak{p}$. We may define the Clifford algebra of $\mathfrak{p}$ and the corresponding spin representation S of $\mathfrak{k}$. Complexifications of $\mathfrak{g}, \mathfrak{k}$ are denoted by $\mathfrak{g}^{\mathbf{C}}, \mathfrak{k}^{\mathbf{C}}$, respectively.

Given a Harish Chandra module $(\pi, \mathcal{H})$ of $(\mathfrak{g}^{\mathbf{C}}, K)$, there is a Dirac operator $\mathcal{D}_\pi : \mathcal{H} \otimes S \to \mathcal{H} \otimes S$ defined by

$$\mathcal{D}_\pi = \sum_j \pi(X_j) \otimes \gamma(X_j)$$

where $\{X_j\}$ is an orthonormal basis of $\mathfrak{p}$ and γ is the Clifford multiplication on S.

J.-S. Huang and P. Pandžić [5] proved a remarkable result which marked a revival of interest in the role of the Dirac operator in representation theory Dirac operator - focus in which was a driving force starting from the 70's. (*eg.* [1],[12],[15],[8],[16],[10],[11] to mention a few - not to mention numerous authors who used Dirac inequality in questions of unitarizability, notably,[17],[18],[19],[20],[21]). This was a proof of a conjecture of Vogan, relating the infinitesimal character of an irreducible Harish Chandra module π and an irreducible $\mathfrak{k}$-type occurring in the kernel of the formal Dirac operator $\mathcal{D}_\pi$, more precisely, not just the kernel but kernel modulo its intersection with the image of $\mathcal{D}$ which is called Dirac cohomology. Kostant who had introduced the cubic Dirac operator([4],[6]) for a wide range of homogeneous spaces G/H more general than the Riemannian symmetric space G/K, realized the potential of the Huang and Pandžić result and obtained an analogous result for the cubic Dirac operator ([7]).

In [9] we were motivated to view in a coherent way how the spaces of (representation theoretic) harmonic spinors vary when the representation parameters change in a coherent way.

We illustrate in the following context: let $F(\nu)$ be the finite dimensional representation of $\mathfrak{g}^{\mathbf{C}}$ (the complexification of $\mathfrak{g}$) with highest weight ν, with respect to some positive system. Let $\{\pi(\mu)\}_{\mu\in\Lambda}$ be a coherent family of (virtual) representations of G (see [2]). Typically in a positive cone contained in the parameter space Λ, this arises via the Zuckerman translation functor of tensoring with an irreducible finite-dimensional module and projecting to the central eigenspace corresponding to a shifted infinitesimal character.

$$F(\nu)^\star \otimes \pi(\mu + \nu) \longrightarrow \pi(\mu)$$

where $F(\nu)^\star$ denotes the contragredient representation of $F(\nu)$.

Denote by $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{W}_3$ the kernel of the Dirac operator associated with $\pi_{\mu+\nu}, F(\nu)^*$ and π_μ respectively.

The problem is to understand how $\mathcal{W}_1, \mathcal{W}_2$ and $\mathcal{W}_3$ are related. For the case of discrete series representations we illustrated this in [9,(Theorem 4.2)]. More precisely, when G has a compact Cartan subgroup, it is known that discrete series representations of G arise as a particular case of Enright-Varadarajan $(\mathfrak{g}^{\mathbf{C}}, K)$-modules ([3][14]), where K still denotes a maximal compact subgroup of maximal rank of G. Moreover, given a $(\mathfrak{g}^{\mathbf{C}}, K)$-module $(\pi, \mathcal{H})$, there is a Dirac operator $\mathcal{D}_\pi : \mathcal{H} \otimes S \to \mathcal{H} \otimes S$ defined by

$$\mathcal{D}_\pi = \sum_j \pi(X_j) \otimes \gamma(X_j)$$

where $\{X_j\}$ is an orthonormal basis of $\mathfrak{p}$ and γ is the Clifford multiplication. Now assume that $(\pi(\mu), \mathcal{H}(\mu))$ and $(\pi(\mu+\nu), \mathcal{H}(\mu+\nu))$ are two discrete series representations of G, regarded as $(\mathfrak{g}^{\mathbf{C}}, K)$-modules, where μ is dominant integral regular and ν is dominant integral with respect to some positive system in $\mathfrak{g}^{\mathbf{C}}$. Standard inclusions of Verma modules for $\mathfrak{g}^{\mathbf{C}}$ give rise to an inclusion $\varphi : \mathcal{H}(\mu) \otimes S \hookrightarrow \mathcal{H}(\mu + \nu) \otimes F(\nu) \otimes S$. Moreover, we have a map $\beta : (\mathcal{H}(\mu+\nu) \otimes S) \otimes (F(\nu) \otimes S) \otimes S^\star \longrightarrow \mathcal{H}(\mu+\nu) \otimes F(\nu) \otimes S$, by contracting the second factor S and the fifth factor $S^\star$. The statement of [9, Theorem 4.2] is:

Theorem: $\varphi(\mathrm{Ker}(\mathcal{D}_{\pi(\mu)})) \subseteq \beta(\mathrm{Ker}(\mathcal{D}_{\pi(\mu+\nu)}) \otimes \mathrm{Ker}(\mathcal{D}_{F(\nu)}) \otimes S^\star)$.

In other words, one can relate Dirac spinors for an irreducible representation, a finite dimensional irreducible representation and a third representation which is related to the first two via a Zuckerman translation. We have used the Enright-Varadarajan construction in the proof of this result for a description of the discrete series representations.

2. θ-stable parabolic subalgebra $\mathfrak{q}$ and generalized Enright-Varadarajan modules.

Recall that we assume that G has finite center and has a compact Cartan subgroup. As we have already mentioned, discrete series representations arise as a particular case of Enright-Varadarajan modules ([3][14]). More generally, the so-called cohomological representations $A_{\mathfrak{q}}(\lambda)$ arise as generalized Enright-Varadarajan modules associated to a θ-stable parabolic subalgebra [13]. Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{r} \subset \mathfrak{q}$ be a θ-stable Borel subalgebra of $\mathfrak{g}$. Fix a θ-stable Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ such that $\mathfrak{c} \subset \mathfrak{r}$ and $\mathfrak{b} \stackrel{def.}{=} \mathfrak{c} \cap \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$. (*Our equal rank assumption in fact implies that* $\mathfrak{b} = \mathfrak{c}$). Let P be the corresponding (θ-stable) system of positive roots. We fix a Borel subalgebra $\mathfrak{r}_{\mathfrak{k}}$ of $\mathfrak{k}$ by $\mathfrak{r}_{\mathfrak{k}} \stackrel{def.}{=} \mathfrak{r} \cap \mathfrak{k}$. The corresponding positive system of roots of $\mathfrak{k}$ with respect to $\mathfrak{b}$ is denoted $P_{\mathfrak{k}}$. We shall denote by δ and $\delta_{\mathfrak{k}}$ the half-sums of P and $P_{\mathfrak{k}}$ respectively. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a Levi decomposition of $\mathfrak{q}$ so that $\mathfrak{c} \subset \mathfrak{l}$ and $\mathfrak{u}$ is the sum of the root spaces in the nilradical of $\mathfrak{q}$. We can write P as a disjoint union $P = P_{\mathfrak{l}} \cup P_{\mathfrak{u}}$, so that $\mathfrak{u}$ is the sum of the root spaces corresponding to roots in $P_{\mathfrak{u}}$. Let μ be a regular integral weight which is dominant with respect to P. Let σ be the unique element of the Weyl group $W_{\mathfrak{g}}$ of $\mathfrak{g}$ such that $\sigma(P) = P_{\mathfrak{l}} \cup -P_{\mathfrak{u}}$. Let $\sigma_{\mathfrak{k}}$ be the unique element of the Weyl group $W_{\mathfrak{k}}$ of $\mathfrak{k}$ such that $\sigma_{\mathfrak{k}}(P_{\mathfrak{k}}) = P_{\mathfrak{l}\cap\mathfrak{k}} \cup -P_{\mathfrak{u}\cap\mathfrak{k}}$. Consider the Verma module $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ for $\mathfrak{g}$ with highest weight $\sigma(\mu) - \delta$ with respect to P and the Verma modules $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma(\mu)-\delta}$ for $\mathfrak{k}$ with $P_{\mathfrak{k}}$-highest weight given by the restriction of $\sigma(\mu) - \delta$ to $\mathfrak{b}$. Evidently, $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma(\mu)-\delta}$ can be canonically identified with the $\mathcal{U}(\mathfrak{k})$-module generated by the highest weight vector of $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$. There is a unique $P_{\mathfrak{k}}$-dominant integral weight η such that $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma(\mu)-\delta} \subseteq V_{\mathfrak{k},P_{\mathfrak{k}},\eta}$. One has $\sigma_{\mathfrak{k}}(\eta + \delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}} = \sigma(\mu) - \delta$. The $\mathfrak{k}$-module $V_{\mathfrak{k},P_{\mathfrak{k}},\eta}$ and the $\mathfrak{g}$-module $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ can both be simultaneously imbedded in a $\mathfrak{g}$-module $W_{\mathfrak{q},\mu}$, compatible with the imbedding $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma(\mu)-\delta} \subseteq V_{\mathfrak{k},P_{\mathfrak{k}},\eta}$ and having nice properties. Some of the important properties of the inclusions $V_{\mathfrak{g},P,\sigma(\mu)-\delta} \subseteq W_{\mathfrak{q},\mu}$ and $V_{\mathfrak{k},P_{\mathfrak{k}},\eta} \subseteq W_{\mathfrak{q},\mu}$ are the following (see [13]):

(i) $W_{\mathfrak{q},\mu}$ has a unique irreducible quotient $\mathfrak{g}$-module $D_{\mathfrak{q},\mu}$ which is $\mathfrak{k}$-finite,

(ii) the irreducible finite dimensional $\mathfrak{k}$-module $F_{\mathfrak{k},\eta}$ with $P_{\mathfrak{k}}$- highest weight η occurs with multiplicity one in $D_{\mathfrak{q},\mu}$,

(iii) if $\chi_{P,-\mu}$ denotes the algebra homomorphism from $U(\mathfrak{g})^{\mathfrak{k}}$ into $\mathbf{C}$ defining the scalar by which $u \in \mathcal{U}(\mathfrak{g})^{\mathfrak{k}}$ acts on the highest weight vector of $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$, then the same homomorphism gives the action of $\mathcal{U}(\mathfrak{g})^{\mathfrak{k}}$ on $F_{\mathfrak{k},\eta} \subseteq D_{\mathfrak{q},\mu}$.

Let $\ell(\cdot)$ denote the length function in $W_{\mathfrak{k}}$ as a minimal product of simple reflections. Choose a reduced expression $\sigma_{\mathfrak{k}} = s_1 \cdot s_2 \cdots s_m$. There is a chain of imbeddings of $\mathfrak{k}$-Verma modules $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{m+1}$ which can be fitted inside the imbedding $V_1 = V_{\mathfrak{k},P_{\mathfrak{k}},\sigma_{\mathfrak{k}}(\eta+\delta_{\mathfrak{k}})-\delta_{\mathfrak{k}}} \subseteq V_{\mathfrak{k},P_{\mathfrak{k}},\eta} = V_{m+1}$.

(iv) The above mentioned $\mathfrak{g}$-module imbedding $V_{\mathfrak{g},P,\sigma(\mu)-\delta} \subseteq W_{\mathfrak{q},\mu}$ can be spread to a chain of $\mathfrak{g}$-module imbeddings $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{m+1}$ enlarging the chain of $\mathfrak{k}$-module imbeddings $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{m+1}$. In other words, $W_1 = V_{\mathfrak{g},P,\sigma(\mu)-\delta}, W_{m+1} = W_{\mathfrak{q},\mu}$ and we have $V_i \subseteq W_i$.

We can now state some additional properties of this chain of $\mathfrak{g}$-modules constructed in [13]. Let ϕ_i be the simple root of $P_{\mathfrak{k}}$ defined so that $s_i = s_{\phi_i}, i = 1, 2, \cdots, m$. Let $\mathfrak{m}_i$ be the 3-dimensional simple Lie algebra generated by the root vectors of $\mathfrak{k}$ corresponding to $\phi_i, i = 1, 2, \cdots, m$.

(v) Each W_i is $\mathcal{U}(\mathfrak{n}_{\mathfrak{k}})$-free; here $\mathfrak{n}_{\mathfrak{k}}$ is the sum of all rootspaces of $\mathfrak{k}$ corresponding to roots in $P_{\mathfrak{k}}$.

(vi) $\frac{W_{i+1}}{W_i}$ is $\mathfrak{m}(\phi_i)$-finite.

The last two properties remain true even when W_i is replaced by $W_i \otimes F$ where F is any finite dimensional $\mathfrak{k}$-module. We will be interested in applying these properties to $W_i \otimes S$, where S is the spin module.

3. Statement of the result.

When G has a compact Cartan subgroup, which we assume to hold, in the special case where $\mathfrak{q} = \mathfrak{r}$, (i.e., the parabolic subalgebra is a Borel subalgebra) it is a result due to Wallach ([12]) that the $(\mathfrak{g}, K)$-module $D_{\mathfrak{r},\mu}$ is isomorphic to the space of $\mathfrak{k}$-finite vectors in a discrete class representation of G. A similar identification can be shown to exist between the more general $D_{\mathfrak{q},\mu}$ and the so-called cohomological representations usually referred to as '$A_{\mathfrak{q}}(\lambda)$'.

(3.1). Let ν be P-dominant integral and $F(\nu)$ the finite dimensional irreducible representation for $\mathfrak{g}$ with highest weight ν. So $\mu+\nu$ is P-dominant and regular and we have the irreducible $(\mathfrak{g}, K)$-modules $D_{\mathfrak{q},\mu}$ and $D_{\mathfrak{q},\mu+\nu}$ as above. We have a canonical inclusion $\varphi : D_{\mathfrak{q},\mu} \hookrightarrow D_{\mathfrak{q},\mu+\nu} \otimes F(\nu)^*$ which is a consequence of the inclusion of Verma modules for $\mathfrak{g}$: $V_{\mathfrak{g},P,\sigma(\mu)-\delta} \hookrightarrow V_{\mathfrak{g},P,\sigma(\mu+\nu)-\delta} \otimes F(\nu)^*$. In turn this gives rise to an inclusion $\varphi_S : D_{\mathfrak{q},\mu} \otimes S \hookrightarrow D_{\mathfrak{q},\mu+\nu} \otimes F(\nu)^* \otimes S$. Moreover, we have a map $\beta : (D_{\mathfrak{q},\mu+\nu} \otimes S) \otimes (F(\nu)^* \otimes S) \otimes S^\star \to D_{\mathfrak{q},\mu+\nu} \otimes F(\nu)^* \otimes S$ by contracting the second factor S and the fifth factor $S^\star$.

Now, we can state our result. Denote by $\mathcal{W}_1^o, \mathcal{W}_2$ and $\mathcal{W}_3^o$ the kernel of the Dirac operator associated with $D_{\mathfrak{q},\mu+\nu}, F(\nu)^*$ and $D_{\mathfrak{q},\mu}$ respectively. Denote by $\mathcal{W}_1$,(resp.$\mathcal{W}_3$) the subspace of $\mathcal{W}_1^o$, (resp.$\mathcal{W}_3^o$), spanned by $\mathfrak{k}$-types of $P_{\mathfrak{k}}$-highest ξ such that $\xi + \delta_{\mathfrak{k}}$ is $W_{\mathfrak{g}}$-conjugate to $\mu + \nu$,(resp.μ).
Remark:- For unitary representations $\mathcal{W}^o = \mathcal{W}$. Folklore has it that probably even for the non-unitary ones in the coherent continuation considered here, this equality may have been noted somewhere in the literature; we have not come across this explictly.

Theorem 1. *We have:* $\varphi_S(\mathcal{W}_3) \subseteq \beta(\mathcal{W}_1 \otimes \mathcal{W}_2 \otimes S^*)$.

Proof:- The method involves relating $\mathcal{W}_1$ and $\mathcal{W}_3$ to the kernel of the Dirac operator acting on $V_{\mathfrak{g},P,\sigma(\mu+\nu)-\delta} \otimes S$ and $V_{\mathfrak{g},P,\sigma(\mu)-\delta} \otimes S$.

It is easy to describe the kernel of the Dirac operator acting on $F(\nu)^* \otimes S$. This kernel is the $\mathcal{U}(\mathfrak{k})$-span of $x \otimes s$ where x is a weight vector of $F(\nu)^*$ of weight $w(-\nu)$ in the Weyl group orbit of $-\nu$ and is $P_{\mathfrak{k}}$-dominant and s is a $P_{\mathfrak{k}}$-highest weight vector of an irreducible component of S of weight $-\delta_{\mathfrak{k}} - w\delta$.

Next, we describe some elements in the kernel of the Dirac operator $\mathcal{D}$ acting on $D_{\mathfrak{q},\mu+\nu} \otimes S$ as in [9]. (Similar remarks for $D_{\mathfrak{q},\mu} \otimes S$ will hold). Later we use "Vogan's Conjecture"{ *now a 'Theorem' (due to Huang and Pandzic) which reads off the infinitesimal character of an irreducible* $(\mathfrak{g}, K)$*-moddule by looking at the* $P_{\mathfrak{k}}$*-highest weight of an irreducible* $\mathfrak{k}$*-type in the kernel of the Dirac operator for that module*} to complement this description and conclude that there is nothing besides the elements in the kernel described this way.

3.2. Let y' be a $P_{\mathfrak{k}}$-highest weight vector of weight γ' in $V_{\mathfrak{g},P,\sigma(\mu+\nu)-\delta} \otimes S$ annihilated by the Dirac operator. Assume that $\gamma' + \delta_{\mathfrak{k}}$ is $\sigma_{\mathfrak{k}}(P_{\mathfrak{k}})$-dominant and non-singular. Let γ be a $P_{\mathfrak{k}}$-dominant integral weight such that $\gamma + \delta_{\mathfrak{k}} \in W_{\mathfrak{k}}(\gamma' + \delta_{\mathfrak{k}})$. Then there is a $P_{\mathfrak{k}}$-highest weight vector $\tilde{y}$ of $W_{\mathfrak{q},\mu+\nu} \otimes S$ of weight γ such that $\mathcal{U}(\mathfrak{k}) \cdot y' \subseteq \mathcal{U}(\mathfrak{k}) \cdot \tilde{y}$. It is not difficult to show that $\tilde{y}$ (hence, also its image y in $D_{\mathfrak{q},\mu+\nu} \otimes S$) is in the kernel of the Dirac operator. We apply these observations by making the simplest choice for y'. Namely, $y' = x \otimes s$, where x is the P-highest weight vector of weight $\sigma(\mu+\nu) - \delta$ of $V_{\mathfrak{g},P,\sigma(\mu+\nu)-\delta}$ and s is the $P_{\mathfrak{k}}$-highest weight vector of S of weight $\delta - \delta_{\mathfrak{k}}$.

3.3. Let $\overline{y}'$ be a $P_{\mathfrak{k}}$-highest weight vector of weight $\overline{\gamma}'$ in $V_{\mathfrak{g},P,\sigma(\mu)-\delta} \otimes S$ annihilated by the Dirac operator. Assume that $\overline{\gamma}' + \delta_{\mathfrak{k}}$ is $\sigma_{\mathfrak{k}}(P_{\mathfrak{k}})$-dominant and non-singular. Let $\overline{\gamma}$ be a $P_{\mathfrak{k}}$-dominant integral weight such that $\overline{\gamma} + \delta_{\mathfrak{k}} \in W_{\mathfrak{k}} \cdot (\overline{\gamma}' + \delta_{\mathfrak{k}})$. Then there is a $P_{\mathfrak{k}}$-highest weight vector $\tilde{\overline{y}}$ of $W_{\mathfrak{q},\mu} \otimes S$ of weight $\overline{\gamma}$ such that $\mathcal{U}(\mathfrak{k}) \cdot \overline{y}' \subseteq \mathcal{U}(\mathfrak{k}) \cdot \tilde{\overline{y}}$. One can show that $\tilde{\overline{y}}$ (hence, also its image $\overline{y}$ in $D_{\mathfrak{q},\mu} \otimes S$) is in the kernel of the Dirac operator. Apply these observations by choosing $\overline{y}' = \overline{x} \otimes s$, where $\overline{x}$ is the P-highest weight vector of weight $\sigma(\mu) - \delta$ of $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ and s is the $P_{\mathfrak{k}}$-highest weight vector of S of weight $\delta - \delta_{\mathfrak{k}}$. Note that if ξ is the $P_{\mathfrak{k}}$-highest weight of this choice of $\overline{y}'$, then $\xi + \delta_{\mathfrak{k}} = \sigma(\mu) \in W_{\mathfrak{g}}(\mu)$.

The statement analogous to result stated in [Theorem 1] relating $(V_{\mathfrak{g},P,\sigma(\mu+\nu)-\delta} \otimes S) \otimes (F^*(\nu) \otimes S) \otimes S^\star$ and $(V_{\mathfrak{g},P,\sigma(\mu)-\delta} \otimes S)$, namely the fact that

(3.4). $\varphi_S(\overline{y}') \in \beta(y' \otimes W_2 \otimes S^\star)$

is evident.

(3.5). Properties (v) and (vi) of the chain $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_m$ and the fact that the maps φ_S and β are restrictions of corresponding maps obtained by changing W_1 to W_m imply (using 3.4) statements analogous to 3.4 relating $(W_{\mathfrak{q},\mu+\nu}\otimes S)\otimes(F^*(\nu)\otimes S)\otimes S^\star$ and $(W_{\mathfrak{q},\mu}\otimes S)$ and these in turn can finally be related to $(D_{\mathfrak{q},\mu+\nu}\otimes S)\otimes(F^*(\nu)\otimes S)\otimes S^\star$ and $(D_{\mathfrak{q},\mu}\otimes S)$.

We recall the last two properties of the chain of $\mathfrak{g}$-modules $W_i, i = 1,2,\cdots,m$

(v) Each W_i is $\mathcal{U}(\mathfrak{n}_\mathfrak{k})$-free; here $\mathfrak{n}_\mathfrak{k}$ is the sum of all rootspaces of $\mathfrak{k}$ corresponding to roots in $P_\mathfrak{k}$.

(vi) $\frac{W_{i+1}}{W_i}$ is $\mathfrak{m}(\phi_i)$-finite.

Having fixed a reduced expression $\sigma_\mathfrak{k} = s_1\cdot s_2\cdots s_m$, define submodules $\overline{W}_1,\overline{W}_2,\cdots,\overline{W}_m,\overline{W}$ and W_X as in 4.3 and 4.5 in [13]. Let $\overline{V}_1\subseteq\overline{V}_2\subseteq\cdots\subseteq\overline{V}_m$ be a chain of Verma modules for $\mathfrak{k}$. Let v_{m+1} be a highest weight vector of $\overline{V}_{m+1}$ of weight μ_{m+1} which is $P_\mathfrak{k}$-dominant integral. Now assume that $\overline{V}_{m+1}\subseteq W_{m+1}$ so that $\overline{V}_i\subseteq W_i\cap\overline{V}_{m+1}$. We have the following important property.

(vii) $\overline{V}_{m+1} = 0 \mod W_X$, if and only if $\overline{V}_1\subseteq\sum V_i'\subseteq W_1\cap\overline{V}_{m+1}$, where each V_i' is a $\mathfrak{k}$-Verma module with highest weight $\sigma_i(\mu_{m+1}+\delta_\mathfrak{k})-\delta_\mathfrak{k}$ and $\ell(\sigma_i) < m$.

The last three properties remain true even when W_i is replaced by $W_i\otimes F$ where F is any finite dimensional $\mathfrak{k}$-module. We will be interested in applying these properties to $W_i\otimes S$, where S is the spin module.

Using (vii) we wish to construct more non-zero elements in the Dirac kernel essentially via constructions similar to 3.2 and 3.3. Recall that $\sigma(P_\mathfrak{l}\cup P_\mathfrak{u}) = \sigma(P) = P_\mathfrak{l}\cup -P_\mathfrak{u}$. Write $P' = P_\mathfrak{l}\cup -P_\mathfrak{u}$. Choose a positive system $\overline{P}_\mathfrak{l}$ in the root system $\Delta_\mathfrak{l} = P_\mathfrak{l}\cup -P_\mathfrak{l}$ such that $\overline{P}_{\mathfrak{l},\mathfrak{k}} = P_{\mathfrak{l},\mathfrak{k}}$. Put $\overline{P} = \overline{P}_\mathfrak{l}\cup P_\mathfrak{u}$. Note that $\overline{P}_\mathfrak{k} = P_\mathfrak{k}$. Put $\overline{P}' = \overline{P}_\mathfrak{l}\cup -P_\mathfrak{u}$. Note that $\overline{P}'_\mathfrak{k} = P'_\mathfrak{k} = \sigma_\mathfrak{k}(P_\mathfrak{k})$ $=\mathrm{P}_{\mathfrak{l},\mathfrak{k}}\cup -P_{\mathfrak{u},\mathfrak{k}}$. Choose $\overline{\sigma}\in W_\mathfrak{g}$ such that $\overline{\sigma}(P) = \overline{P}'$. The crucial observation is that the irreducible quotients (as $\mathfrak{g}$-modules) of $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ and $V_{\mathfrak{g},\overline{P},\overline{\sigma}(\mu)-\overline{\delta}}$ are isomorphic. Both are isomorphic to $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{q})}W$, where W is the irreducible $\mathfrak{l}$-module whose highest weight with respect to $P_\mathfrak{l}$ is $\sigma(\mu)-\delta = \sigma(\mu-\delta)-2\delta_\mathfrak{u}$. Then the highest weight of W with respect to $\overline{P}_\mathfrak{l}$ is $\overline{\sigma}(\mu)-\overline{\delta}$ which equals $\overline{\sigma}(\mu-\delta)-2\delta_\mathfrak{u}$. We have a surjection $V_{\mathfrak{l},P_\mathfrak{l},\sigma(\mu)-\delta}\longrightarrow W$. By lemma 9 in [3] the $P_{\mathfrak{l},\mathfrak{k}}$ highest weight vector $\hat{v}_\mathfrak{l}$ of weight $\overline{\sigma}(\mu-\delta)-2\delta_\mathfrak{u}$ in W can be pulled back to a $P_{\mathfrak{l},\mathfrak{k}}$ highest weight vector $\overline{v}_\mathfrak{l}$ of $V_{\mathfrak{l},P_\mathfrak{l},\sigma(\mu)-\delta}$. We identify $V_{\mathfrak{l},P_\mathfrak{l},\sigma(\mu)-\delta} = \{v\in V_{\mathfrak{g},P,\sigma(\mu)-\delta}\mid X_\alpha\cdot v = 0,\forall\alpha\in P_\mathfrak{u}\}$. Then, $X_\alpha\cdot\overline{v}_\mathfrak{l} = 0,\forall\alpha\in P_\mathfrak{k}$.

Note that $\overline{\delta} - \delta_{\mathfrak{k}}$ is the highest weight of an irreducible $\mathfrak{k}$-type in the spin module S. Let $s_{\overline{\delta}-\delta_{\mathfrak{k}}}$ be a corresponding highest weight vector. By following the construction outlined in **3.3** taking $\overline{y}' = \overline{v}_{\mathfrak{l}} \otimes s_{\overline{\delta}-\delta_{\mathfrak{k}}}$ we get more elements in the kernel of the Dirac operator $\mathcal{D}_{\mathfrak{q},\mu}$ on $D_{\mathfrak{q},\mu} \otimes S$, one for each choice of $\overline{P}_{\mathfrak{l}}$ as above.
Remark: For notational convenience, we denote the Dirac operator on $W_{\mathfrak{q},\mu} \otimes S$ also by $\mathcal{D}_{\mathfrak{q},\mu}$. We do not need a separate notation for the Dirac operator on $W_1 \otimes S$ as this is just the restriction of $\mathcal{D}_{\mathfrak{q},\mu}$.
Further replacing μ by $\mu + \nu$ we get similar elements in the kernel of the Dirac operator $\mathcal{D}_{\mathfrak{q},\mu+\nu}$ on $D_{\mathfrak{q},\mu+\nu} \otimes S$ as in **3.2** , one for each choice of $\overline{P}_{\mathfrak{l}}$ and a statement analogous to 3.4. Using (vii) it can be deduced that all these elements are non-zero. Note that if ξ is the $P_{\mathfrak{k}}$-highest weight of this choice of $\overline{y}'$, then $\xi + \delta_{\mathfrak{k}} = \overline{\sigma}(\mu) \in W_{\mathfrak{g}}(\mu)$. Thus all the elements constructed thus in $\mathcal{W}^o$ in fact belong to the subspace $\mathcal{W}$.

(3.6). Next, we will briefly indicate how from Vogan's conjecture relating Dirac kernel and infinitesimal character we can conclude that there are'nt more elements in $\mathcal{W}$ than what is in the linear span of the ones indicated above.

Let $z \in D_{\mathfrak{q},\mu} \otimes S$ and suppose that z is in the Dirac kernel and that z is a non-zero $P_{\mathfrak{k}}$-highest weight vector of weight ξ. Since Vogan's conjecture is a theorem, we know that $\xi + \delta_{\mathfrak{k}} \in W_{\mathfrak{g}} \cdot \mu$. By lemma 9 in [3] the $P_{\mathfrak{k}}$-highest weight vector $z \in D_{\mathfrak{q},\mu} \otimes S$ of weight ξ can be pulled back to a $P_{\mathfrak{k}}$-highest weight vector $\overline{z} \in W_{\mathfrak{q},\mu} \otimes S$ of weight ξ. We observe that $\overline{z} \neq 0 \bmod W_X$, while, $\mathcal{D}_{\mathfrak{q},\mu}\overline{z} = 0 \bmod W_X \otimes S$. We apply property (vii) to $W_i \otimes S$. The intersection of the $\mathfrak{k}$-Verma module generated by $\overline{z}$ and $W_1 \otimes S (= V_{\mathfrak{g},P,\sigma(\mu)-\delta} \otimes S)$ contains the $\mathfrak{k}$-Verma module $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma_{\mathfrak{k}}(\xi+\delta_{\mathfrak{k}})-\delta_{\mathfrak{k}}}$ which is non-zero mod $W_0 \otimes S$ and furthermore $\mathcal{D}_{\mathfrak{q},\mu}(V_{\mathfrak{k},P_{\mathfrak{k}},\sigma_{\mathfrak{k}}(\xi+\delta_{\mathfrak{k}})-\delta_{\mathfrak{k}}}) = 0 \bmod W_0 \otimes S$. We fix a non-zero highest weight vector $\tilde{z}$ of $V_{\mathfrak{k},P_{\mathfrak{k}},\sigma_{\mathfrak{k}}(\xi+\delta_{\mathfrak{k}})-\delta_{\mathfrak{k}}}$ of weight $\sigma_{\mathfrak{k}}(\xi + \delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}}$.

We know that the weights of the $\mathfrak{g}$-Verma module $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ are of the form $\sigma(\mu) - \delta - \sum_{\alpha \in P} m_\alpha \alpha$, where m_α are non-negative integers. Similary, the weights of the $\mathfrak{g}$-Verma module $V_{\mathfrak{g},\overline{P},\overline{\sigma}(\mu)-\overline{\delta}}$ are of the form $\overline{\sigma}(\mu) - \overline{\delta} - \sum_{\alpha \in \overline{P}} m_\alpha \alpha$, where m_α are non-negative integers. The weights of the spin module S (a $\mathfrak{k}$-module) are of the form $\delta - \delta_{\mathfrak{k}} - < A >|_{\mathfrak{c}_{\mathfrak{k}}}$ for certain subsets $A \subseteq P$ disjoint from $P_{\mathfrak{k}}$ and $< A > = \sum_{\alpha \in A} \alpha$. For the same reason, for the positive systems $\overline{P}$ considered above (in particular, $\overline{P}_{\mathfrak{k}} = P_{\mathfrak{k}}$) the weights of the spin module S are also of the form $\overline{\delta} - \overline{\delta}_{\mathfrak{k}} - < A >|_{\mathfrak{c}_{\mathfrak{k}}}$ for certain subsets $A \subseteq \overline{P}$ disjoint from $P_{\mathfrak{k}}$. Both $V_{\mathfrak{g},P,\sigma(\mu)-\delta}$ and $V_{\mathfrak{g},\overline{P},\overline{\sigma}(\mu)-\overline{\delta}}$ have the same irreducible quotient, namely, $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{q})}W$. Since $\tilde{z}\neq 0 \bmod W_0$, $\tilde{z}$ has a non-zero image in $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{q})}W\otimes S$.

(3.7). These remarks imply that if β is the weight of $\tilde{z}$ then $\beta+\delta_{\mathfrak{k}}$ is of the form $\overline{\sigma}(\mu)-\sum_{\alpha\in\overline{P}}m_\alpha\alpha- < A >$, where $m_\alpha(\alpha\in\overline{P})$ are non-negative integers and $A\subseteq\overline{P}$ is a subset disjoint from $P_{\mathfrak{k}}$ for any of the positive systems $\overline{P}$ considered above. Furthermore, $\beta+\delta_{\mathfrak{k}}=\sigma_{\mathfrak{k}}(\xi+\delta_{\mathfrak{k}})$.

(3.8.) We already observed that as a consequence of Vogan's conjecture $\xi+\delta_{\mathfrak{k}}\in W_{\mathfrak{g}}\cdot\mu$. So, $\beta+\delta_{\mathfrak{k}}\in W_{\mathfrak{g}}\cdot\mu$. Since $\xi+\delta_{\mathfrak{k}}$ is dominant regular with respect to $P_{\mathfrak{k}}$ and since $\sigma_{\mathfrak{k}}(P_{\mathfrak{k}})=P_{\mathfrak{l},\mathfrak{k}}\cup -P_{\mathfrak{u},\mathfrak{k}}$, we note that $\beta+\delta_{\mathfrak{k}}$ is dominant regular with respect to $P_{\mathfrak{l},\mathfrak{k}}\cup -P_{\mathfrak{u},\mathfrak{k}}$. Let $\overline{P}_{\mathfrak{l}}$ be the positive system in $P_{\mathfrak{l}}\cup -P_{\mathfrak{l}}$ such that $\beta+\delta_{\mathfrak{k}}$ is dominant regular with respect to $\overline{P}_{\mathfrak{l}}$. Define two positive systems $\overline{P}$ and Q as follows: as earlier, $\overline{P}=\overline{P}_{\mathfrak{l}}\cup P_{\mathfrak{u}}$, $Q=\{\alpha\in(P\cup -P)\mid \beta+\delta_{\mathfrak{k}}(\alpha)>0\}$. Again, following earlier notation, let $\overline{\sigma}\in W_{\mathfrak{g}}$ be the unique element such that $\overline{\sigma}(P)=\overline{P}_{\mathfrak{l}}\cup -P_{\mathfrak{u}}$. The basic point is: $\overline{\sigma}(\mu)$ is dominant regular with respect to $\overline{P}_{\mathfrak{l}}\cup -P_{\mathfrak{u}}$, $\beta+\delta_{\mathfrak{k}}$ is dominant regular with respect to Q and they are in the same $W_{\mathfrak{g}}$ orbit. This situation forces $\beta+\delta_{\mathfrak{k}}=\overline{\sigma}(\mu)$ + a negative integral combination of $\overline{\sigma}(P)\cap -Q$. From our remarks in the beginning of this paragraph about where all $\beta+\delta_{\mathfrak{k}}$ is positive it is clear that $\overline{\sigma}(P)\cap -Q$ is disjoint from $P_{\mathfrak{k}}$ and $P_{\mathfrak{l}}\cup -P_{\mathfrak{l}}$. So, $\beta+\delta_{\mathfrak{k}}=\overline{\sigma}(\mu)$ + a non-negative integral combination of roots in $P_{\mathfrak{u}}$. Comparing this with 3.6, the non-negative integral combination in the last sentence is null and the quantity $-\sum_{\alpha\in\overline{P}}m_\alpha\alpha- < A >$ in 3.6 is also null, ending the discussion begun in 3.6.

REFERENCES

[1] M. Atiyah and W. Schmid, *A geometric construction of the discrete series for semisimple Lie groups*, Inv. Math. **42**(1977), 1–62

[2] D. Barbasch and D. Vogan, *Weyl group representations and nilpotent orbits*, in Representation theory of reductive groups. Progr. Math. **40**, pp. 21-33, Birkhäuser, 1983

[3] T. J. Enright and V. S. Varadarajan, *On an infinitesimal characterization of the discrete series*, Ann. of Math. **102**(1975), 1–15

[4] B. Gross, B. Kostant, P. Ramond and S. Sternberg, *The Weyl character formula, the half-spin representations, and equal rank subgroups*, Proc. Natl. Acad. Sci. USA **95**(1998), 8441–8442

[5] J.-S. Huang and P. Pandžić, *Dirac cohomology, unitary representations and a proof of a conjecture of Vogan*, J. Amer. Math. Soc. **15**(2002), 185–202

[6] B. Kostant, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. J. **100**(1999), 447–501

[7] B. Kostant, *Dirac cohomology for the cubic Dirac operator*, Studies in Memory of Issia Schur (Chevaleret/Rehovot, 2000). Progr. Math. **210**, 69–93, Birkhäuser, 2003

[8] G. D. Landweber, *Harmonic spinors on homogeneous spaces*, Representation Theory **4**(2000), 466–473

[9] S. Mehdi and R. Parthasarathy, *A Product for Harmonic Spinors on Reductive Homogeneous Spaces*, Journal of Lie Theory **18**(2008), 33–44

[10] S. Mehdi and R. Zierau, *Harmonic spinors on semisimple symmetric spaces*, J. Funct. Anal. **198**(2003), 536–557

[11] S. Mehdi and R. Zierau, *Principal series representations and harmonic spinors*, Adv. Math. **1999**(2006), 1–28

[12] R. Parthasarathy, *Dirac operator and discrete series*, Ann. of Math. **96**(1972), 1–30

[13] R. Parthasarathy, *A generalization of the Enright-Varadarajan modules*, Compositio Math. **36** (1978)

[14] N. R. Wallach, *On the Enright-Varadarajan modules: a construction of the discrete series*, Ann. Sci. Ecole. Norm. Sup. (4) **9**(1976), 81–101

[15] J. A. Wolf, *Partially harmonic spinors and representations of reductive Lie groups*, J. Funct. Anal. **15**(1974), 117–154

[16] S. Slebarski, *The Dirac operator on homogeneous spaces and representations of reductive Lie groups II*, Amer. J. Math., **109** (1987), 499-520.

[17] R. Parthasarathy, *Criteria for the unitarizability of some highest weight modules*, Proc. Ind. Acad. Sci., **89** (1980), 1-24.

[18] T. J. Enright, R. Parthasarathy, *A proof of a conjecture of Kashiwara and Vergne*, “Non-commutative Harmonic Analysis and Lie Groups”, Lecture Notes in Mathematics, **880** Springer-Verlag, (1981), 74-90.

[19] S. Kumaresan, *On the canonical k-types in the irreducible unitary g-modules with non-zero relative Lie algebra cohomology*, Inventiones Math., **59** (1980), 1-11.

[20] D. A. Vogan, *Unitarizability of certain series of representations*, Ann. of Math. (2) **120** (1984), no. 1, 141–187.

[21] D. A. Vogan, G. J. Zuckerman, *Unitary representations with nonzero cohomology*, Compositio Math. **53** (1984), no. 1, 51–90.

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