Off-shell representations
of the super Poincaré group

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The super Poincaré group

Recall that the Poincaré group,

$$P^n := \mathrm{Spin}(1, n-1) \ltimes \mathbb{R}^{1,n-1},$$

is the double cover of the group of isometries of Minkowski space. This is the fundamental symmetry group of relativistic physics, consisting of spacetime translations, rotations, and Lorentz boosts.

The super Poincaré group is its supersymmetric extension,

$$P^{n|s} := \mathrm{Spin}(1, n-1) \ltimes \left(\mathbb{R}^{1,n-1} \times \Pi\mathbb{S}\right),$$

by a real spin 1/2 representation $\mathbb{S}$ of $\mathrm{Spin}(1, n-1)$. Here Π is the parity reversal operator, indicating that $\Pi\mathbb{S}$ is odd, and $s = \dim \mathbb{S}$.

The super vector space $\mathbb{R}^{n|s} := \mathbb{R}^{1,n-1} \times \Pi\mathbb{S}$ is Salam-Strathdee superspace, a supersymmetric extension of Minkowski spacetime.

Superspace is Nonabelian

At the Lie superalgebra level, the anti-commutator of two infinitesimal odd translations $s, t \in \Pi\mathbb{S}$ is

$$\{s,t\} = \{t,s\} = 2\,\partial_{\Gamma(s,t)},$$

where

$$\Gamma : \mathbb{S} \times \mathbb{S} \longrightarrow \mathbb{R}^{1,n-1}$$

is a $\mathrm{Spin}(1, n-1)$-equivariant *symmetric* bilinear form (which exists for any real spin representation in Minkowski signature).

The super Poincaré group is often viewed as the super group of isometries of superspace. However, this is not entirely true. The factor $\mathbb{S}$ in the super Poincaré group consists of only the left (or right) superspace translations. The translations of superspace give two such factors $\mathbb{S}_L \times \mathbb{S}_R$, just as for a nonabelian Lie group G we have the doubled group $G \times G$ of left- $\times$ right-translations.

On-shell Representations — Wigner's Construction

The $\mathbb{R}^{1,n-1}$ factor is abelian, so its irreducible representations are one dimensional, parametrized by covectors $\xi \in (\mathbb{R}^{1,n-1})^*$.

The $\mathrm{Spin}(1, n-1)$ factor acts on $(\mathbb{R}^{1,n-1})^*$, so we must take the direct integral of such representations along a covector orbit

$$\mathcal{O} = \mathrm{Spin}(1, n-1)\,\xi \subset (\mathbb{R}^{1,n-1})^*,$$

called a *mass shell*. Furthermore, we have

$$\mathcal{O} \cong \mathrm{Spin}(1, n-1)/\,\mathrm{Stab}(\xi),$$

where $\mathrm{Stab}(\xi)$ is the *little group*, acting on the representation at ξ.

This gives us a $\mathrm{Spin}(1, n-1)$-homogeneous vector bundle $E \to \mathcal{O}$, and the corresponding *on-shell* representation of P^n is $L^2(E)$. Such representations correspond to representations of $\mathrm{Stab}(\xi)$.

Off-shell Representations

In contrast, *off-shell representations* of P^n are not restricted to a mass shell, instead consisting of fields defined on all of $(\mathbb{R}^{1,n-1})^*$.

Via a Fourier transform, off-shell representations are of the form

$$L^2(\mathbb{R}^{1,n-1}) \otimes V,$$

where V is a finite dimensional representation of $\mathrm{Spin}(1, n-1)$.

Off-shell representations are the basic representations used in physics. Equations of motion, such as the Klein-Gordon or Dirac equations, restrict an off-shell representation to the on-shell sub-representation of fields supported on a single mass shell. This way, a single off-shell representation restricts to all massive and massless shells, and one can also restrict to the non-physical zero and tachyon shells. These shells have little groups $\mathrm{Spin}(n-1)$, $\mathrm{Spin}(n-2)$, $\mathrm{Spin}(1, n-1)$, and $\mathrm{Spin}(1, n-2)$, respectively.

Supersymmetric Version

Odd generators $s, t \in \Pi\mathbb{S}$ act on momentum space fields $\phi(\xi)$ with

$$\big((st + ts)\phi\big)(\xi) = 2i\,\xi\big(\Gamma(s,t)\big)\phi(\xi).$$

However, this is a Clifford relation for the symmetric bilinear form

$$i\xi \circ \Gamma : \mathbb{S} \times \mathbb{S} \to \mathbb{C}.$$

On-shell representations of $P^{n|s}$ are spaces of sections of $\mathbb{Z}_2$-graded homogeneous bundles over mass shells, with little group replaced by

$$\mathrm{Stab}(\xi) \ltimes \mathrm{Cl}(\mathbb{S}, i\xi \circ \Gamma).$$

Off-shell representations of $P^{n|s}$ are spaces of fields on all of $\mathbb{R}^{1,n-1}$ or $(\mathbb{R}^{1,n-1})^*$, taking values in a representation V of $\mathrm{Spin}(1, n-1)$, with a $\mathrm{Spin}(1, n-1)$-equivariant family of Clifford actions of $\mathrm{Cl}(\mathbb{S}, i\xi \circ \Gamma)$ on V, parameterized by *all* covectors $\xi \in (\mathbb{R}^{1,n-1})^*$.

The Off-shell Supersymmetry Problem

Problem: Classify all off-shell representations of the super Poincaré group $P^{n|s}$, up to isomorphism.

Off-shell representations are known only for $n \leq 6$, constructed using superfields defined on superspace. String and M-theory need 10 and 11 dimensions, where we have only on-shell supersymmetry.

Off-shell supersymmetry separates representation theory from the physics (i.e., the Lagrangian), and off-shell theories are easier to quantize: we quantize a linear space and then reduce, rather than reducing first and then trying to quantize a nonlinear space.

Jim Gates' "fundamental supersymmetry challenge" is to determine the off-shell extensions of known on-shell theories.

This simple problem has remained unsolved for over 30 years!

Units and Engineering Dimension

Fields, like all physical quantities, possess *units*.

Fixing physical constants allows us to identify units, such as setting $c = 1$ identifies distance and time. We cannot eliminate all units, but we can express everything in terms of a single unit, say mass, which is usually measured in (electron volts)$/c^2$.

Denoting our units of mass by $[M]$, all units can be transformed into units $[M]^n$ where n is the *mass* or *engineering dimension*.

For example, time and distance have engineering dimension -1 and are measured in units of $[M]^{-1}$. So, a spacetime derivative takes units $[M]^n$ to $[M]^{n+1}$, increasing engineering dimension by 1.

The odd generators of the super Poincaré group are "square roots" of derivatives, so they increase the engineering dimension by $1/2$.

Formal Off-shell Representations

Multiplying engineering dimension by 2, the super Poincaré algebra $\mathfrak{p}^{n|s}$ becomes a $\mathbb{Z}$-graded superalgebra, decomposing by degree as

$$\mathfrak{p}^{n|s} = \mathfrak{spin}(1, n-1)_{(0)} \oplus \Pi\mathbb{S}_{(1)} \oplus \mathbb{R}^{1,n-1}{}_{(2)}.$$

Definition. A *formal off-shell representation* of $\mathfrak{p}^{n|s}$ is a $\mathbb{Z}$-graded representation, which is also a finitely generated, free module over

$$U\left(\mathbb{R}^{1,n-1}\right) \cong \mathrm{Sym}^*\left(\mathbb{R}^{1,n-1}\right)$$

for the abelian Lie subalgebra $\mathbb{R}^{1,n-1} \subset \mathfrak{p}^{n|s}$.

This definition replaces the Hilbert space of momentum space fields

$$L^2\left((\mathbb{R}^{1,n-1})^*\right) \otimes V$$

with the $\mathbb{Z}$-graded and dense subspace of polynomial fields

$$\mathrm{Sym}^*(\mathbb{R}^{1,n-1}) \otimes V.$$

One-Dimensional Supersymmetry

We now focus our attention on $P^{1|N}$, also known as N-extended, $d = 1$ supersymmetry, used in supersymmetric quantum mechanics.

Here, $\mathrm{Spin}(1,0) \cong \mathbb{Z}_2$, identified with the $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$, and the super Poincaré algebra is the $\mathbb{Z}$-graded algebra

$$\mathfrak{p}^{1|N} := \mathrm{Span}_{\mathbb{R}}\{Q_1, \ldots, Q_N\}_{(1)} \oplus \mathbb{R}H_{(2)},$$

with non-vanishing brackets

$$\{Q_i, Q_j\} = 2\,\delta_{ij}\,H.$$

Compare this with the *Clifford algebra* $\mathrm{Cl}(N)$, the unital superalgebra with odd generators $\gamma_1, \ldots, \gamma_N$ and relations

$$\{\gamma_i, \gamma_j\} = 2\,\delta_{ij}.$$

Filtered Clifford Supermodules

The Clifford algebra is only $\mathbb{Z}_2$-graded, but it has an increasing $\mathbb{Z}$-filtration given by assigning the generators γ_i filtration degree 1.

Identifying H with 1 in the super Poincaré algebra, we obtain

$$U(\mathfrak{p}^{1|N}) \,/\, \langle H - 1 \rangle \cong \mathrm{Cl}(N),$$

as an isomorphism of $\mathbb{Z}_2$-graded and $\mathbb{Z}$-filtered algebras.

Also, for a formal off-shell representation $\mathcal{V}$ of $\mathfrak{p}^{1|N}$, the quotient

$$\mathcal{V} \,/\, \langle H - 1 \rangle \mathcal{V}$$

is a $\mathbb{Z}_2$-graded and $\mathbb{Z}$-filtered module over $\mathrm{Cl}(N)$.

Theorem. *There is a one-to-one correspondence between formal off-shell representations of* $\mathfrak{p}^{1|N}$ *and* $\mathbb{Z}$*-filtered* $\mathrm{Cl}(N)$*-supermodules.*

Formal Deformations and Rees Spaces

Proof. Given a $\mathbb{Z}_2$-graded and $\mathbb{Z}$-filtered vector space V, its formal deformation (see Gerstenhaber, 1966) or Rees space is

$$\operatorname{Def}_t V := \Big\{ \sum\nolimits_{p=0}^{k} v_p\, t^p \;\Big|\; v_p \in F_p V \cap V_{p \bmod 2} \Big\},$$

which is $\mathbb{Z}$-graded according to the powers of the parameter t.

(Ignoring the $\mathbb{Z}_2$-grading, the formal deformation satisfies

$$(\operatorname{Def}_t V)_p = F_p V,$$

transforming the $\mathbb{Z}$-filtration on V into a $\mathbb{Z}$-grading on $\operatorname{Def}_t V$.)

The formal deformation is the inverse to the quotient by $\langle H - 1\rangle$:

$$V \cong \operatorname{Def}_t V \,/\, \langle t^2 - 1\rangle \operatorname{Def}_t V,$$
$$\mathcal{V} \cong \operatorname{Def}_t\big(\mathcal{V} \,/\, \langle H - 1\rangle \mathcal{V}\big).$$

A One-Parameter Family of Clifford Actions

In our case, identifying $Q \mapsto t\gamma_i$ and $H \mapsto t^2$ gives an isomorphism

$$U(\mathfrak{p}^{1|N}) \cong \mathrm{Def}_t\,\mathrm{Cl}(N),$$

and Def_t takes filtered $\mathrm{Cl}(N)$-supermodules to $U(\mathfrak{p}^{1|N})$-modules.

Alternatively, the formal deformation of a $\mathbb{Z}_2$-graded and $\mathbb{Z}$-filtered vector space V gives rise to a 1-parameter family of vector spaces:

$$\mathrm{Def}_\lambda V = \mathrm{Def}_t V \,/\, (t^2 - \lambda)\,\mathrm{Def}_t V.$$

In our case, the family $\mathrm{Def}_\xi V$ for a $\mathbb{Z}$-filtered $\mathrm{Cl}(N)$-supermodule V possesses a one-parameter family of Clifford actions

$$\mathrm{Def}_\xi\,\mathrm{Cl}(N) \cong \mathrm{Cl}(\mathbb{S}, \xi \circ \Gamma),$$

which precisely specifies an off-shell representation of $P^{1|N}$. $\square$

Bifiltered Clifford Supermodules

For two dimensional supersymmetry, we have

$$\mathrm{Spin}(1,1) \cong \mathbb{Z}_2 \times \mathbb{R}^{>0},$$

giving us a second integral grading, with decompositions:

$$\mathbb{R}^{1,1} \cong \mathbb{R}_{(+2)} \oplus \mathbb{R}_{(-2)}, \quad \mathbb{S}^s = \mathbb{S}^p_{(+1)} \oplus \mathbb{S}^q_{(-1)},$$

into light-cone coordinates and chiral spinors.

Formal off-shell representations of $\mathfrak{p}^{2|p,q}$, with (p,q)-supersymmetry, are then equivalent to bigraded representations of $\mathfrak{p}^{1|p} \oplus \mathfrak{p}^{1|q}$.

Theorem. *Formal off-shell representations of* $\mathfrak{p}^{2|p,q}$ *are precisely the formal deformations of bifiltered* $\mathrm{Cl}(p+q)$*-supermodules, where*

$$\mathrm{Cl}(p+q) := \mathrm{Cl}(p) \,\widetilde{\otimes}\, \mathrm{Cl}(q),$$

with bifiltration induced from the filtrations on the two factors.

Higher Dimensions

The higher dimensional generalization of these statements to n-filtered Clifford supermodules violates Lorentz invariance. However, we are working to prove a significant partial result:

Theorem. *Let* $\mathbb{S}$ *be a real spin representation of* $\mathrm{Spin}(1, n-1)$*, and fix a positive energy timelike (massive) covector* $\xi \in (\mathbb{R}^{1,n-1})^*$*. Let* V *be a filtered* $\mathrm{Cl}(\mathbb{S}, \xi \circ \Gamma)$*-supermodule satisfying*

- V *admits a unitary* $\mathrm{Spin}(1, n-1)$*-action, such that*
- *for each* $s \in \mathbb{S}$*, the Clifford action* $c(s)$ *is self-adjoint, and*
- *the associated graded action* $\mathrm{gr}\, c(s)$ *is* $\mathrm{Spin}(1, n-1)$*-equivariant.*

Then V *extends uniquely to an off-shell representation of* $\mathfrak{p}^{n|s}$*.*

(The inner product on V and the self adjoint Clifford action allow us to construct a supersymmetric Lagrangian, and the associated graded condition is obtained by restricting to the zero shell.)

Extending $d = 2$ Off-shell Representations

Alternatively, using equivariantly bifiltered Clifford supermodules:

Theorem. *Fixing a pair of complementary lightlike (massless) covectors* $\xi_L, \xi_R \in (\mathbb{R}^{1,n-1})^*$*, a real spin representation of* $\mathrm{Spin}(1, n-1)$ *decomposes as* $\mathbb{S} = \mathbb{S}_L \oplus \mathbb{S}_R$ *with respect to* $\mathrm{Spin}(n-2) = \mathrm{Stab}(\xi_L, \xi_R)$*. Let* V *be a bifiltered supermodule over*

$$\mathrm{Cl}\big(\mathbb{S}, (\xi_L + \xi_R) \circ \Gamma\big) \cong \mathrm{Cl}(\mathbb{S}_L, \xi_L \circ \Gamma) \mathbin{\widetilde{\otimes}} \mathrm{Cl}(\mathbb{S}_R, \xi_R \circ \Gamma),$$

with a $\mathrm{Spin}(1, n-1)$*-action satisfying*

- *the bifiltration is* $\mathrm{Spin}(n-2)$*-equivariant,*
- *the Clifford action* $c(s)$ *is* $\mathrm{Spin}(n-1)$*-invarant,*
- *the associated graded action* $\mathrm{gr}\, c(s)$ *is* $\mathrm{Spin}(1, n-1)$*-equivariant,*
- *there is an involution* $V \to V$ *interchanging the two filtrations.*

Then V *extends uniquely to an off-shell representation of* $\mathfrak{p}^{n|s}$*.*

References

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