Fock model for minimal representations

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The oscillator representation is a unitary representation of the metaplectic group, which is a covering of order 2 of the symplectic group $Sp(n, \mathbb{R})$.

It can be realized on $L^2(\mathbb{R}^n)$: the Schrödinger model,

or on the Fock space $\mathcal{F}(\mathbb{C}^n)$, the space of holomorphic functions on $\mathbb{C}^n$ which are square integrable with respect to a Gaussian measure: the Fock model.

Analogues of the oscillateur representation for non Hermitian simple real Lie groups has been considered in many papers.

Among them:
Brylinski-Kostant for the analogue of the Fock model,
Kobayashi-Ørsted, Kobayashi-Mano for the analogue of the Schrödinger model.

In this talk we revisit the construction by Brylinski-Kostant.

V is a complex Euclidean vector space. Q is a polynomial on V, homogeneous of degree 4.

$$L = \{g \in GL(V) \mid \exists \gamma(g) \in \mathbb{C},\ Q(g \cdot z) = \gamma(g) Q(z)\}.$$

One assumes that L has an open orbit, and L is selfadjoint : for $g \in L$, $g^* \in L$ also.

More precisely V is a semi-simple complex Jordan algebra of rank ≤ 4, L is the structure group of V, and Q is a semi-invariant for L of degree 4.

K is the conformal group of V: $g \in K$ is a rational transformation of V such that, at every $z \in V$ where g is defined,

$$Dg(z) \in L.$$

The translations are conformal transformations, and $L \subset K$.

The inversion $\sigma(z) = -z^{-1}$ is a conformal transformation. Up to a factor $\sigma(z) = \nabla \log Q$.

K is a semi-simple Lie group, and $P = L \ltimes V$ is a maximal parabolic subgroup of K.

$\mathfrak{p}$ is the vector space generated by the translated of Q: the polynomials

$$p_a(z) = Q(z-a) \quad (a \in V).$$

Let us first assume that there is a character χ_0 of L such that

$$Q(g \cdot z) = \chi_0(g)^2 Q(z).$$

Then the conformal group K acts on $\mathfrak{p}$ by:

$$\Big(\kappa(g)p\Big)(z) = \alpha(g,z)p(g^{-1} \cdot z),$$

where

$$\alpha(g^{-1},z) = \chi_0\Big(Dg(z)\Big)^{-1}.$$

In particular

$$\begin{aligned}\Big(\kappa(\tau_a)p\Big)(z) &= p(z-a) \quad (a \in V),\\ \Big(\kappa(\ell)p\Big)(z) &= \chi_0(\ell)p(\ell^{-1}z) \quad (\ell \in L),\\ \Big(\kappa(\sigma)p\Big)(z) &= Q(z)p(-z^{-1}).\end{aligned}$$

If such a character χ_0 does not exist, one considers a covering $\tilde{K}$ of K of order 2, and defines similarly a representation κ of $\tilde{K}$ on $\mathfrak{p}$.

Example 1

1. $V = \mathbb{C}$, $Q(z) = z^4$, $L = \mathbb{C}^*$, $K = PSL(2, \mathbb{C})$ acting on $\mathbb{C}$ by the fractional linear transformations:

$$g \cdot z = \frac{az + b}{cz + d}.$$

$\mathfrak{p}$ is the space of polynomials in one variable of degree ≤ 4,

$$\Big(\kappa(g)\Big)(z) = (cz + d)^4 p\Big(\frac{az + b}{cz + d}\Big),$$

if $g^{-1} \cdot z = \frac{az+b}{cz+d}$.

Example 2

$V = M(4, \mathbb{C})$, $Q(z) = \det z$,
$L = GL(4, \mathbb{C}) \times GL(4, \mathbb{C})/\mathbb{C}^*$,
$K = SL(8, \mathbb{C})/\{\pm I\}$, with the action

$$g \cdot z = (az + b)(cz + d)^{-1}.$$

$$\Big(\kappa(g)p\Big)(z) = \det(cz+d)p\Big((az+b)(cz+d)^{-1}\Big),$$

if $g^{-1} \cdot z = (az + b)(cz + d)^{-1}$.

Graduation of $\mathfrak{k}$ and $\mathfrak{p}$

$\mathfrak{k} = Lie(K)$ is the so called Kantor-Koecher-Tits Lie algebra of the Jordan algebra V.

h_t dilation of V: $h_t \cdot z = e^{-t} z \quad (t \in \mathbb{R})$
$h_t \in L$, $h_t = e^{tH}$, $H \in Lie(L)$, $\chi_0(h_t) = e^{-2t}$.

H defines a graduation of $\mathfrak{k}$:

$$\mathfrak{k} = \mathfrak{k}_{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1,$$
$$\mathfrak{k}_j = \{X \in \mathfrak{k} \mid \mathrm{ad}(H)X = jX\} \quad (j = -1, 0, 1).$$

H defines a graduation of $\mathfrak{p}$. $d\kappa(H) = \mathcal{E} - 2$, $\mathcal{E}$ is the Euler operator

$$(\mathcal{E}p)(z) = \langle z, \nabla p(z) \rangle.$$

$$\mathfrak{p} = \mathfrak{p}_{-2} \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2,$$
$$\mathfrak{p}_j = \{p \in \mathfrak{p} \mid d\kappa(H)p = jp\}.$$

$\mathfrak{p}_j$: homogeneous polynomials of degree $j+2$ in $\mathfrak{p}$.

$\mathfrak{p}_{-2} = \mathbb{C}$, $\mathfrak{p}_2 = \mathbb{C}Q$.

$\kappa(\sigma) : \mathfrak{p}_j \to \mathfrak{p}_{-j}$.

Generalized Kantor-Koecher-Tits construction

Define $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, as a vector space.
Notation: $E = Q$, $F = 1$ $(E, F \in \mathfrak{p})$.

Theorem *There is on* $\mathfrak{g}$ *a unique Lie algebra structure for which*

$$[X, p] = d\kappa(X)p \quad (X \in \mathfrak{k},\ p \in \mathfrak{p}),$$
$$[E, F] = H.$$

A direct proof has been obtained by D. Achab, but it turned out that is has been established a long time ago in a more algebraic setting by Allison and Faulkner (1984).

Orbit Ξ

Ξ is the K-orbit of Q in $\mathfrak{p}$,

$$\Xi = \{\kappa(g)Q \mid g \in K\} \subset \mathfrak{p}.$$

The conformal group K acts on the space $\mathcal{O}(\Xi)$ of holomorphic functions on Ξ:

$$\Big(\pi(g)f\Big)(\xi) = f\Big(\kappa(g^{-1})\xi\Big).$$

Ξ is a conical manifold: there is an action of $\mathbb{C}^*$ on Ξ which commutes with the action of K.

$\mathcal{O}_m(\Xi)$: space of holomorphic functions which are homogeneous of degree $m \in \mathbb{Z}$:

$$f(\lambda\xi) = \lambda^m f(\xi) \quad (\lambda \in \mathbb{C}^*).$$

Proposition $\mathcal{O}_m(\Xi) = \{0\}$ *for* $m < 0$. $\mathcal{O}_m(\Xi)$ *is finite dimensional, invariant, and irreducible under the action of* K.

Every $f \in \mathcal{O}(\Xi)$ can be written

$$f(\xi) = \sum_{m=0}^{\infty} f_m(\xi) \quad (f_m \in \mathcal{O}_m(\Xi)).$$

The series converges uniformly on compact sets.

Coordinates on Ξ_0

A polynom $\xi \in \mathfrak{p}$ can be written

$\xi(v) = wQ(v)+$terms of degree $< 4 \quad (w \in \mathbb{C})$,

and $\xi \mapsto w = w(\xi)$ is a linear form on $\mathfrak{p}$. Define

$$\Xi_0 = \{\xi \in \Xi \mid w(\xi) \neq 0\}.$$

This is a dense open set in Ξ. A polynom $\xi \in \Xi_0$ can be written

$$\xi(v) = wQ(v-z) \quad (z \in V).$$

One gets in that way coordinates $(z, w) \in V \times \mathbb{C}^*$ for Ξ_0.

The expansion

$$f(\xi) = \sum_{m=0}^{\infty} f_m(\xi)$$

takes the form, when restricted to Ξ_0,

$$f(z, w) = \sum_{m=0}^{\infty} \psi_m(z) w^m.$$

Representation ρ of $\mathfrak{g}$

Following a method due to R. Brylinski and B. Kostant, we construct a representation ρ of $\mathfrak{g}$, whose restriction to $\mathfrak{k}$ is $d\pi$, on

$$\mathcal{O}_{\mathrm{fin}}(\Xi) = \sum_{m=0}^{\infty} \mathcal{O}_m(\Xi) \quad \text{(finite sums).}$$

In the variables $(z, w) \in V \times \mathbb{C}^*$ define the operators

$$(Mf)(z,w) = wf(z,w),$$
$$(Df)(z,w) = \frac{1}{w}\Big(Q\big(\frac{\partial}{\partial z}\big)f\Big)(z,w).$$

and a diagonal operator

$$\delta : \sum_m f_m \mapsto \sum_m \delta_m f_m,$$

where (δ_m) is a sequence of numbers, to be determined.

One constructs first a representation of $\mathfrak{sl}(2, \mathbb{C})$. One puts

$$\rho(F) = M - D \circ \delta.$$

Since $E = \kappa(\sigma)F = \mathrm{Ad}(\sigma)F$, necessarily

$$\rho(E) = M^\sigma - D^\sigma \circ \delta,$$

where $M^\sigma = \pi(\sigma)M\pi(\sigma)$, $D^\sigma = \pi(\sigma)D\pi(\sigma)$. One checks easily that

$$[\rho(H), \rho(E)] = 2\rho(E), \quad [\rho(H), \rho(F)] = -2\rho(F).$$

Then the sequence (δ_m) is determined by the condition

$$[\rho(E), \rho(F)] = \rho(H).$$

Theorem *Assume that the type of* $\mathfrak{g}$ *is* A, B, D, *or* E, *and, in case*

$$\mathfrak{k} = \mathfrak{so}(p+2, \mathbb{C}) \times \mathfrak{so}(q+2, \mathbb{C}),$$
$$\mathfrak{g} = \mathfrak{so}(p+q+4, \mathbb{C}),$$

assume that $p = q$. *For*

$$\delta_m = \frac{A}{(m+c)(m+c+1)},$$

where A, c *are constants depending on* (V, Q),

$$[\rho(E), \rho(F)] = \rho(H),$$

and one gets a representation ρ *of* $\mathfrak{g}$ *on* $\mathcal{O}_{\mathsf{fin}}(\Xi)$.

For the proof, one uses

- $\mathcal{O}_m(\Xi)$ decomposes multiplicity free under $K_0 = L$. This follows from the Schmid decomposition.
- Diagonalisation of the Maass operators for a simple Jordan algebra:

$$D_\alpha = \det(x)^{\alpha+1} \det\Big(\frac{\partial}{\partial x}\Big) \det(x)^{-\alpha}.$$

- Selberg formula: if $D_\alpha^\sigma f = D_\alpha(f \circ \sigma) \circ \sigma$,

$$D_\alpha^\sigma = D_\beta,$$

with $\beta = \frac{n}{r} - 1 - \alpha$ ($n = \dim V, r = \mathrm{rank} V$).

Hilbert structures

One fixes a compact real form $K_{\mathbb{R}}$ of the conformal group K. There is a unique $K_{\mathbb{R}}$-invariant Hilbert structure on $\mathcal{O}_m(\Xi)$ such that $\|w^m\| = 1$.
Let $\Phi(\xi, \eta)$ be the reproducing kernel of $\mathcal{O}_1(\Xi)$. then one shows that the reproducing kernel of $\mathcal{O}_m(\Xi)$ equals $\Phi(\xi, \eta)^m$.

Theorem *The group* $K_{\mathbb{R}}$ *acts multiplicity free on* $\mathcal{O}(\Xi)$*. This means that every* $K_{\mathbb{R}}$*-invariant Hilbert subspace* $\mathcal{H} \subset \mathcal{O}(\Xi)$ *decomposes without multiplicity:*

$$\mathcal{H} = \bigoplus_{m \in \Lambda} \mathcal{O}_m(\Xi) \quad (\Lambda \subset \mathbb{N}).$$

The reproducing kernel $R(\xi, \eta)$ *of* $\mathcal{H}$ *is given by*

$$R(\xi, \eta) = \sum_{m \in \Lambda} c_m \Phi(\xi, \eta)^m.$$

The series converges uniformly on compact sets.

One considers a real simple Lie group $G_{\mathbb{R}}$ such $Lie(G_{\mathbb{R}})$ is a real form of $\mathfrak{g}$, and $K_{\mathbb{R}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$. Up to coverings:

The sum $E + F$ belongs to $\mathfrak{g}_{\mathbb{R}}$. One determines a Hilbert subspace $\mathcal{F}(\Xi) \subset \mathcal{O}(\Xi)$ which is $K_{\mathbb{R}}$-invariant, and for which $\rho(E+F)$ is essentially skewsymmetric.

It amounts to determining a sequence (c_m).

Theorem *There is a unitary representation T of $G_{\mathbb{R}}$ on $\mathcal{F}(\Xi)$ such that $dT = \rho$. The representation T is spherical.*

For $\mathcal{F}(\Xi)$ the c_m are coefficients of a hypergeometric function ${}_2F_3$:

$$\begin{aligned}
&{}_2F_3(\alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3; x) \\
&= \sum_{m=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_m}{(\beta_1)_m (\beta_2)_m (\beta_3)_m} \frac{1}{m!} x^m \\
&= \sum_{m=0}^{\infty} c_m x^m.
\end{aligned}$$

The parameters α_i, β_i depend on (V, Q).

The reproducing kernel of $\mathcal{F}(\Xi)$ is given by

$$R(\xi, \eta) = {}_2F_3\Big(\alpha_1, \alpha_2; \beta_1, \beta_2, \beta_3; \Phi(\xi, \eta)\Big).$$

Minimal representations

Let $\mathcal{O}$ be a nilpotent coadjoint orbit of $\mathfrak{g}$, and π an irreducible unitary representation of $G_{\mathbb{R}}$ on a Hilbert space $\mathcal{H}$. This representation defines a representation of the universal envelopping algebra of $\mathfrak{g}$ on $\mathcal{H}^{\infty}$. Let $\mathcal{J}$ be its kernel, and grad$(\mathcal{J})$ the associated graded ideal. One says that the the representation π is associated to the orbit $\mathcal{O}$ if $\mathcal{V}(\text{grad}(\mathcal{J})) = \overline{\mathcal{O}}$. The representation is said to be minimal if it is associated to the minimal nilpotent coadjoint orbit $\mathcal{O}_{\text{min}}$.

One shows that

$$\Xi = \mathcal{O}_{\mathsf{min}} \cap \mathfrak{p}.$$

In case of type A, the representation T is minimal if n is even ≥ 2.

In cases B, D, and E, T is minimal.