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STABLE COMBINATIONS OF SPECIAL UNIPOTENT REPRESENTATIONS

1. Introduction

This is joint work with Peter Trapa. It relies heavily on [ABV] and on work of Rossmann and Kashiwara-Tanisaki.

2. Representation theory

2.1. **Strong real forms.** Let G be a connected linear algebraic reductive group with Lie algebra $\mathfrak{g}$. We denote by $\Gamma := Gal(\mathbb{C}/\mathbb{R})$ the Galois group of $\mathbb{R}$. A *weak extended group* G^Γ is a (real Lie) group which contains G as a subgroup of order 2, and such that any $\delta \in G^\Gamma \setminus G$ acts by an antiholomorphic automorphism (definition 2.13 in [ABV]). A classification of weak extended groups is at the end of chapter 2 of [ABV], and the more refined notion of *extended group* is in definition 1.12 of [ABV].

The group G corresponds to the root datum $\Phi = (\mathcal{X}, \mathcal{Y}, R, {}^\vee R)$, and δ induces an automorphism $a \in Aut(\Phi)$ via $\operatorname{Ad}\delta$.

Definition (2.13 and 2.14 in [ABV]). *A strong real form of G^Γ is an element $\delta \in G^\Gamma \setminus G$ such that $\delta^2 \in Z(G)$ is of finite order. Denote by $G(\mathbb{R}, \delta)$ the fixed points of δ on G. A representation of a strong real form of G^Γ is a pair (δ, π) where δ is a strong real form, and π is an admissible representation of $G(\delta, \mathbb{R})$. Two representations (δ, π) and (δ', π') are equivalent if there exists $g \in G$ such that $\delta' = g\delta g^{-1}$, and $\pi \cong \pi' \circ \operatorname{Ad}(g^{-1})$ (i.e. are infinitesimally equivalent). Write $\Pi(G/\mathbb{R})$ for the set of equivalence classes of irreducible representations of strong real forms of G^Γ.*

2.2. **Example 1.** Assume $G = SL(2)$, and G^Γ contains a $\delta_0 \in G^\Gamma \setminus G$ such that $\operatorname{Ad}\delta_0(x) = \overline{x}$. There are three conjugacy classes of strong real forms, represented by δ_+, δ_-, and δ_0 where

$$\delta_\pm = \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \delta_0. \tag{2.2.1}$$

The corresponding fixed points are to be thought of as $SU(2,0)$, $SU(0,2)$, and $SU(1,1)$ (same as $SL(2,\mathbb{R})$).

When $G = SL(2)/\{\pm I\}$, the same analysis applies, but now $\delta_+ = \delta_-$ and there is only one strong real form instead of the two labeled $SU(2,0)$ and $SU(0,2)$ in the first example.

One should think of the first example as $Sp(2)$ and the second one as $SO(3)$.

2.3. **Strongly stable combinations.** Consider (suitably finite) formal sums

$$\eta = \sum c_{\delta_j, \pi_j} (\delta_j, \pi_j)$$

of elements in $\Pi(G/\mathbb{R})$. Now fix a strong real form δ, and choose irreducible representations $\pi_1, \ldots, \pi_k$ of $G(\mathbb{R}, \delta)$ representing each of the (nonzero) terms in η for strong real forms equivalent to δ. Consider the corresponding generalized function of $G(\mathbb{R}, \delta)$,

$$\Theta(\eta, \delta) := \sum_{i=1}^{k} \Theta_{\pi_i}$$

where Θ_{π_i} is the character of π_i. The original sum η is called *strongly stable* if for any two strong real forms δ and δ' and for any strongly regular $g \in G(\delta, \mathbb{R}) \cap G(\delta', \mathbb{R})$,

$$\Theta(\eta, \delta)(g) = \Theta(\eta, \delta')(g). \tag{2.3.1}$$

This is a refinement of the Langlands-Shelstad notion of stability.

2.4. **Example 2.** Consider the example of $G = SL(2)$ from before, and suppose that η contains the term (δ_0, π_+) where π_+ is the holomorphic discrete series. The condition to be strongly stable, and the existence of a strongly regular $x \in G(\delta_0, \mathbb{R}) \cap G(\delta_+, \mathbb{R})$ and $x' \in G(\delta_0, \mathbb{R}) \cap G(\delta_-, \mathbb{R})$, imply that η must contain the finite dimensional representations with the same infinitesimal character as π_+ and appropriate sign. The requirement to be strongly stable then also implies that η has to contain the antiholomorphic π_- with the same infinitesimal character as π_+, and with the same coefficient.

2.5. **Covering groups.** The reference is section 10 of [ABV]. A finite cover

$$1 \longrightarrow F \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1 \tag{2.5.1}$$

of G is called *distinguished* if

- **(a):** for every $\delta \in G^\Gamma \setminus G$ the automorphism $\sigma_\delta := \operatorname{Ad} \delta$ lifts to $\widetilde{G}$,
- **(b):** the induced automorphism on $Z(\widetilde{G})$ (same for all δ) is $\sigma_Z(z) = z^{-1}$.

The projective limit of all distinguished covers is called G^{can}, and fits in an exact sequence

$$1 \longrightarrow \pi_1(G)^{can} \longrightarrow G^{can} \longrightarrow G \longrightarrow 1 \tag{2.5.2}$$

We will consider representations of the inverse image $G(\mathbb{R}, \delta)^{can}$ of a $G(\mathbb{R}, \delta)$ in G^{can} with a fixed action χ of $\pi_1(G)^{can}$. Here we may use the standard fact that

$$\operatorname{Hom}[\pi_1(G), \mathbb{C}] \cong Z({}^\vee G), \tag{2.5.3}$$

where ${}^\vee G$ is the *dual* group corresponding to the root system ${}^\vee\Phi := (\mathcal{Y}, \mathcal{X}, {}^\vee R, R)$red. Thus we may think of $\chi = \chi_z$ as parametrized by $z \in Z({}^\vee G)$. For each z, we may thus talk about the corresponding equivalence class of irreducible representations $\Pi^z(G/\mathbb{R})$.

Remark. $\pi_1(G) \neq \pi_1(G)^{can}$ in general. To be precise z is really supposed to be an element of $Z({}^\vee G)^\theta$ as in [ABV] chapter 4.

Example. *Let $G = SL(2)/\pm I$. In this case ${}^\vee G = SL(2)$, so the choices of z are $z = \pm I$. Let $\delta = \delta_0$ as in example (2) of section 2.3. Then $G(\mathbb{R}, \delta)^{can} = SL_\pm(2, \mathbb{R})$. The representations $\Pi^z(G/\mathbb{R})$ with $z = -I$ are representations of $SL_\pm(2, \mathbb{R})$ for which the center acts nontrivially.*

2.6. **Parametrization of Π^z.** Recall the E-group ${}^\vee G^\Gamma := {}^\vee G \cdot \{1, {}^\vee\delta\}$ in definition 4.3 of [ABV]. Here ${}^\vee\delta$ is an element satisfying ${}^\vee\delta^2 = z$ and $\mathrm{Ad}\,{}^\vee\delta$ acts on ${}^\vee\Phi$ by ${}^\vee a$, the dual automorphism to the a specified by G^Γ. Let also ${}^\vee G^{alg}$ be the algebraic universal cover of ${}^\vee G$.

Let $\mathcal{O} \subset {}^\vee \mathfrak{g}$ be the orbit of a semisimple element $\lambda \in {}^\vee \mathfrak{g}$. This defines an infinitesimal character, and we denote $\Pi^z(\mathcal{O}, G/\mathbb{R})$ the subset of elements of $\Pi^z(G/\mathbb{R})$ with this infinitesimal character. Let $e(\lambda) := e^{2i\pi\lambda}$; its conjugacy class in ${}^\vee G$ only depends on the orbit $\mathcal{O}$. We denote

$$\begin{aligned} &{}^\vee\mathfrak{g}(\lambda)_i := \{x \in {}^\vee \mathfrak{g} \ : \ [\lambda, x] = ix\}, \\ &\mathcal{F}(\lambda) := \lambda + \sum_{i \in \mathbb{N}^+} {}^\vee\mathfrak{g}(\lambda)_i, \\ &\mathcal{F}(\mathcal{O}) := \{\mathcal{F}(\lambda) \ : \ \lambda \in \mathcal{O}\}, \\ &X(\mathcal{O}, G) := \{(y, \Lambda) \ : \ y \in {}^\vee G^\Gamma, \Lambda \in \mathcal{F}(\mathcal{O}),\ y^2 = e(\Lambda)\}. \end{aligned} \tag{2.6.1}$$

Then ${}^\vee G^{alg}$ acts on $X(\mathcal{O}, G)$ with finitely many orbits. We denote by $\Xi(\mathcal{O}, G)$ (or $\Xi^z(\mathcal{O}, G)$ if we want to emphasize the z), the ${}^\vee G^{alg}$-equivariant local systems on $X(\mathcal{O}, G)$. These parametrize the irreducible objects in the category $\mathcal{P}(\mathcal{O}, G)$ of ${}^\vee G^{alg}$-equivariant perverse sheaves on $X(\mathcal{O}, G)$. There are two bases, equivariant constructible sheaves, and perverse sheaves.

Theorem. *[*[ABV]*, but really a paraphrase of the work of dav@math.mit.edu] There is a perfect pairing between the Grothendieck groups $\mathbb{Z}\mathcal{P}(\mathcal{O}, G)$ and $\mathbb{Z}\Pi^z(\mathcal{O}, G/\mathbb{R})$*

Note: The pairing has the property that constructible sheaves form a dual basis to the standard modules in the sense of the Langlands-Vogan classification, and perverse sheaves are dual to irreducible Langlands subquotients.

The structure of X is as follows. Let

$${}^\vee G(\mathcal{O}) := {}^\vee G(e(\lambda)), \quad {}^\vee P(\mathcal{O}) := {}^\vee P(\lambda), \quad {}^\vee\mathcal{P}(\mathcal{O}) := {}^\vee G(\mathcal{O})/{}^\vee P(\mathcal{O}).$$

Let $y_1, \ldots, y_r$ be representatives of ${}^\vee G$ orbits of the y in the definition of X. Then

$$X(\mathcal{O}, G) = \sqcup_{i=1}^r \ {}^\vee\mathcal{P}(\mathcal{O})_i,$$

each with an action of ${}^\vee K_i := Cent(y_i, {}^\vee\mathfrak{g}(\mathcal{O}))$, and Ξ is the disjoint union of the K_i^{alg}-equivariant local systems.

3. (Special) unipotent representations

An orbit $\mathcal{O}$ of a semisimple element λ defines an infinitesimal character χ_λ. There is a unique maximal primitive ideal $J_{max}(\mathcal{O}) \subset U(\mathfrak{g})$ with infinitesimal character λ, *i.e.* containing the ideal generated by $\{z - \chi_\lambda(z)\}_{z \in U(\mathfrak{g})^G}$.

Definition. *Let* ${}^\vee\mathbb{O} \subset {}^\vee\mathfrak{g}$ *be a nilpotent orbit with corresponding Lie triple* ${}^\vee e, {}^\vee h, {}^\vee f$. *We say that* π *is special unipotent attached to* ${}^\vee\mathbb{O}$ *if it has annihilator equal to* $J_{max}({}^\vee G \cdot {}^\vee h/2)$.

We will give a description of stable combinations of special unipotent representations in $\Pi^z(G, \mathbb{R})$.

4. Stable combinations

By theorem 2.6, representations are interpreted as linear functionals on $\mathbb{Z}\Xi$. Theorem 1.29 in [ABV] identifies which linear functionals correspond to strongly stable combinations.

Local combinations. Let $Q \subset X$ be an orbit, fix $x \in Q$, and denote by T_Q^* its conormal bundle. Define

$$\begin{aligned} m_{Q,loc}(\mathcal{S}) &:= \dim \mathcal{S}_x, && \text{for a constructible sheaf,} \\ m_{Q,loc}(\mathcal{P}) &:= \sum (-1)^i \dim \mathrm{H}^i\, \mathcal{P}_x && \text{for a perverse sheaf.} \end{aligned} \tag{4.0.2}$$

A linear functional is called *geometrically stable* if it is in the $\mathbb{Z}$-span of the $m_{Q,loc}$. The point of theorem 1.29 of [ABV] is that these linear functionals correspond to the Langlands-Shelstad combination of standard modules.

Microlocal combinations. The characteristic cycle provides another basis of the stable linear functionals. Specialize to a single ${}^\vee\mathcal{P}_i$ and view the orbits as K_i orbits. The characteristic cycle is a linear map

$$CC : \mathbb{Z}\Xi_i \longrightarrow \mathbb{Z}\{T_Q^*\}. \tag{4.0.3}$$

So given Q, define

$$m_{Q,mic}(\mathcal{S}) := \text{ coefficient of } T_Q^* \text{ in } CC(\mathcal{S}). \tag{4.0.4}$$

One of the main results of [ABV] is that this is a basis of the space of strongly stable functionals on $\mathbb{Z}\Xi$ (equivalently strongly stable combinations of characters).

5. Main Theorem

Assume that ${}^\vee\mathbb{O}$ is even so that $e({}^\vee h/2)$ is central and ${}^\vee G(e(\lambda)) = {}^\vee G$.

Theorem (Barbasch-Trapa)**.** *There is a canonical basis of the space of strongly stable linear combinations of unipotent characters of representations in* $\Pi^z(\mathcal{O}, G/\mathbb{R})$ *parametrized by* ${}^\vee K_i-$*real forms of nilpotent orbits in* ${}^\vee\mathfrak{g}$ *in the "special piece" associated to* ${}^\vee\mathbb{O}$.

Here

- "special orbit" is in the sense of Lusztig. In particular since the orbit ${}^\vee\mathbb{O}$ is even, it is special.
- An orbit ${}^\vee\mathbb{O}'$ is in the special piece associated to ${}^\vee\mathbb{O}$ if it is in the closure of ${}^\vee\mathbb{O}$ but not in the closure of any special orbit of smaller dimension.

5.1. **Idea of Proof.** Recall from section 2.6 that $\mathbb{Z}\Xi$ is a direct sum of $\mathbb{Z}\Xi_i$ which are in duality with $\mathbb{Z}\Pi_i$. Let ${}^\vee\mathcal{B}_i$ be the flag variety with the action of ${}^\vee K_i$. Using translation functors we can replace $\mathbb{Z}\Pi_i$ with a corresponding Grothendieck group of representations at regular infinitesimal character $\mathbb{Z}\Pi_i^{reg}$. There is an isomorphism

$$\mathbb{Z}\Pi_{i,stable}^{reg} \longleftrightarrow [\mathbb{Z}\mathcal{D}_{{}^\vee\mathcal{B}_i}/\ker CC]^*. \tag{5.1.1}$$

By results of Rossmann or Kashiwara-Tanisaki,

$$[\mathbb{Z}\mathcal{D}_{{}^\vee\mathcal{B}_i}/\ker CC] \cong H_{top}[T^*_{{}^\vee K_i}{}^\vee\mathcal{B}_i, \mathbb{Z}] \tag{5.1.2}$$

As a Weyl group module,

$$H_{top}[T^*_{{}^\vee K_i}{}^\vee\mathcal{B}_i, \mathbb{Z}] = \sum_\xi {}^\vee\mathcal{B}_i(\xi)^{A_{{}^\vee K_i}(\xi)}, \tag{5.1.3}$$

where ξ runs through ${}^\vee K_i-$conjugacy classes of nilpotent elements in ${}^\vee\mathfrak{s}_i$ with ${}^\vee\mathfrak{g} = {}^\vee\mathfrak{k}_i + {}^\vee\mathfrak{s}_i$ the Cartan decomposition, and

$$A_{{}^\vee K_i}(\xi) := Cent[\xi, {}^\vee K_i]/({}^\vee K_i \cap Cent[\xi, {}^\vee G]_0).$$

Let $V_{\phi,\xi}$ be the representation attached to $\phi \in \widehat{A}_{{}^\vee G}(\xi)$ by Springer. Then 5.1.3 becomes

$$\sum [\phi \mid_{A_{{}^\vee K_i}(\xi)} : triv] V_{\phi,\xi}. \tag{5.1.4}$$

The proof follows from the fact that the dimension of $\mathbb{Z}\Pi^{su}_{i,stable}$ is given by

$$\Big[\sum_{\xi \in SP({}^\vee\mathbb{O})} V_{triv,\xi} \ : \ H_{top}[T^*_{{}^\vee K_i}{}^\vee\mathcal{B}_i, \mathbb{Z}] \ \Big]. \tag{5.1.5}$$

This result has the following consequence. Recall that $X(\mathcal{O}, G) = \sqcup_{i=1}^r {}^\vee\mathcal{P}(\mathcal{O})_i$, each variety with an action of ${}^\vee K_i$. The moment map is

$$\mu : T^*_Q{}^\vee\mathcal{P} \longrightarrow {}^\vee\mathfrak{s}_i. \tag{5.1.6}$$

The image is formed of nilpotent elements.

Theorem. *For each ${}^\vee K_i$ orbit ${}^\vee\mathbb{O}'(\mathbb{R})$ of ${}^\vee\mathbb{O}' \cap {}^\vee\mathfrak{s}_i$ in the special piece of ${}^\vee\mathbb{O}$ there is a unique ${}^\vee K_i$ orbit $Q({}^\vee\mathbb{O}'(\mathbb{R})) \subset^\vee \mathcal{P}$ such that $\mu(T^*_Q{}^\vee\mathcal{P})$ is dense in ${}^\vee\mathbb{O}' \cap {}^\vee\mathfrak{s}_i$.*

This is a corollary of the main result. There is a generalization to the non even case.

6. Examples

6.1. **Example 3.** Let $G = SL(2)$ and ${}^\vee G = SL(2)/\pm I$. Assume ${}^\vee\mathbb{O} = (0) \subset {}^\vee\mathfrak{g}$. Then $\lambda = (0)$. The special unipotent representations are $\pi_+,\ \pi_-,\ Ind_B^{G(\delta_0,\mathbb{R})}(triv)$, where $\pi_\pm$ are the two limits of discrete series. The matchup between rational forms and stable combinations is

$$\begin{aligned} (0) \subset so(2,1) &\quad\longleftrightarrow\quad \pi_+ + \pi_-,\\ (0) \subset so(3) &\quad\longleftrightarrow\quad Ind_B^{G(\delta_0,\mathbb{R})}(triv). \end{aligned} \tag{6.1.1}$$

The orbit ${}^\vee\mathbb{O} = (2) \subset {}^\vee\mathfrak{g}$ has only one real form in $so(2,1)$ It corresponds to the strongly stable combination $triv_{SL(2,\mathbb{R})} + triv_{SU(2,0)} + triv_{SU(0,2)}$.

6.2. **Example 4.** Let $G = SL(2)/\pm I$ so that ${}^\vee G = SL(2)$. There are two choices of $z = \pm I$. When ${}^\vee\mathbb{O} = (0)$, we have

$$\begin{aligned} (0) \subset sl(2,\mathbb{R}) &\quad\longleftrightarrow\quad \text{spherical principal series of } SO(2,1)\\ (0) \subset su(2,0),\ su(0,2) &\quad\longleftrightarrow\quad \text{nonspherical principal series of } SL_\pm(2,\mathbb{R}). \end{aligned} \tag{6.2.1}$$

When ${}^\vee\mathbb{O} = (2)$ there are two real forms in $sl(2,\mathbb{R})$. They correspond to the sums $triv^\epsilon_{2,1} + triv_{3,0}$. Here $\epsilon = \pm$ refers to the fact that there are two finite dimensional representations of $SO(2,1)$ with trivial infinitesimal character.

6.3. **Example 5.** Let $G = Sp(4)$ so that ${}^\vee G = SO(5)$, and ${}^\vee\mathbb{O} = (311)$. The special piece attached to ${}^\vee\mathbb{O}$ contains (221) as well. According to [ABV] section 27, there are 8 unipotent representations in $Sp(4,\mathbb{R})$,

$$\begin{aligned} &\pi^1_{++},\ \pi^{-1}_{++}\\ &\pi^1_{+-},\ \pi^{-1}_{+-}\\ &\pi^1_{--},\ \pi^{-1}_{--}\\ &\widetilde{\pi}^1_{+-},\ \widetilde{\pi}^{-1}_{+-} \end{aligned} \tag{6.3.1}$$

and one more in $Sp(2,2)$ labelled $\pi_{2,2}$. The stable combinations are

$$\begin{aligned} \begin{array}{lll} + & - & + \\ + & & \\ + & & \end{array} &\quad\longleftrightarrow\quad \widetilde{\pi}^1_{+-} + \widetilde{\pi}^{-1}_{+-}\\ \begin{array}{lll} + & - & + \\ + & & \\ - & & \end{array} &\quad\longleftrightarrow\quad (\pi^1_{++} + \pi^{-1}_{++}) + (\pi^1_{--} + \pi^{-1}_{--}) + \pi_{2,2}\\ \begin{array}{lll} - & + & - \\ + & & \\ + & & \end{array} &\quad\longleftrightarrow\quad \pi^1_{+-} + \pi^{-1}_{+-}\\ \begin{array}{ll} + & - \\ - & + \\ + & \end{array} &\quad\longleftrightarrow\quad (\pi^1_{++} - \pi^{-1}_{++}) + (\pi^1_{+-} - \pi^{-1}_{+-}) + (\pi^1_{--} - \pi^{-1}_{--}). \end{aligned} \tag{6.3.2}$$

References

[ABV] J. Adams, D. Barbasch, and D. A. Vogan, Jr., *The Langlands Classification and Irreducible Characters for Real Reductive Groups*, Progress in Math, Birkhäuser (Boston), **104**(1992).