Conference
Representations of Lie Groups and Applications
(IHP, Paris, 15–18 December 2008)

Evolution equations in negative curvature

Jean–Philippe Anker
(Université d'Orléans)

December 17, 2008

3 evolution equations

Heat equation

$$\begin{cases} \partial_t u(t,x) - \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f(x) \end{cases}$$

Schrödinger equation

$$\begin{cases} i\,\partial_t u(t,x) \pm \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f(x) \end{cases}$$

Wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f_0(x),\ \partial_t\big|_{t=0} u(t,x) = f_1(x) \end{cases}$$

3 evolution equations

- homogeneous $\rightsquigarrow$ linear case : $g=0$
- possible nonlinearities : $g(t,x)=\Gamma(u(t,x))$

  e.g. $\Gamma(u)=\begin{cases} u\,|u|^{2\sigma} \\ |u|^{1+2\sigma} \end{cases}$ with $\sigma>0$

Symmetric spaces

type	constant curvature	rank one	general case
Euclidean	$\mathbb{R}^n$, $\mathbb{T}^n$	$\mathbb{R}^{(n)}_{\text{(radial)}}$, $\mathbb{T}$	$\mathfrak{p} \rtimes K/K$
compact	$\mathbb{S}^{n-1} = \mathsf{S}(\mathbb{R}^n)$	$\mathsf{S}(\mathbb{F}^n)$	U/K
non compact	$\mathbb{H}^n = \mathsf{H}^n(\mathbb{R})$	$\mathsf{H}^n(\mathbb{F})$	G/K
p–adic		trees	buildings

Symmetric spaces

Other interesting cases :

- Cartan–Hadamard manifolds
- Lie groups $\begin{cases} \text{polynomial growth (compact, nilpotent)} \\ \text{semisimple} \\ \text{amenable (solvable) exponential growth} \end{cases}$
- locally symmetric spaces

Heat equation on symmetric spaces

Heat equation

$$\begin{cases} \partial_t \, u(t,x) - \Delta_x \, u(t,x) = g(t,x) \\ u(0,x) = f(x) \end{cases}$$

Fourier transform :

$$\begin{cases} \partial_t \, \widehat{u}(t,\lambda) + (|\lambda|^2 + |\rho|^2) \, \widehat{u}(t,\lambda) = \widehat{g}(t,\lambda) \\ \widehat{u}(0,\lambda) = \widehat{f}(\lambda) \end{cases}$$

Solution :

$$\widehat{u}(t,\lambda) = \underbrace{e^{-(|\rho|^2+|\lambda|^2)\,t} \, \widehat{f}(\lambda)}_{\text{homogeneous}} + \underbrace{\int_0^t ds \; e^{-(|\rho|^2+|\lambda|^2)\,(t-s)} \, \widehat{g}(s,\lambda)}_{\text{inhomogeneous}}$$

Heat equation on symmetric spaces

Heat equation

$$\begin{cases} \partial_t u(t,x) - \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f(x) \end{cases}$$

Solution

$$u(t,x) = \underbrace{e^{t\Delta_x} f(x)}_{\text{homogeneous}} + \underbrace{\int_0^t ds\, e^{(t-s)\Delta_x} g(s,x)}_{\text{inhomogeneous}}$$

Homogeneous solution :

$$u(t,x) = e^{t\Delta_x} f(x) = f * \underbrace{h_t}_{\text{heat kernel}}(x)$$

Heat kernel on $\mathbb{H}^n$

Explicit expression for $\mathbb{H}^n$ [Debiard–Gaveau–Mazet 1976]

$$h_t(r) = 2^{-\frac{n+1}{2}}\,\pi^{-\frac{n}{2}}\,t^{-\frac{1}{2}}\,e^{-(\frac{n-1}{2})^2 t}\,\Bigl(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\Bigr)^{\frac{n-1}{2}}\,e^{-\frac{r^2}{4\,t}}$$

where

$$\Bigl(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\Bigr)^{\frac{n-1}{2}} f(r) = \frac{1}{\sqrt{\pi}}\int_r^{+\infty} ds\,\frac{\sinh s}{\sqrt{\cosh s - \cosh r}}\,\Bigl(-\frac{1}{\sinh s}\frac{\partial}{\partial s}\Bigr)^{\frac{n}{2}} f(s)$$

if n is even.

Similar expression for

- all hyperbolic spaces $\mathrm{H}^n(\mathbb{F}) = \mathrm{H}^n(\mathbb{R})$, $\mathrm{H}^n(\mathbb{C})$, $\mathrm{H}^n(\mathbb{H})$, $\mathrm{H}^2(\mathbb{O})$ [Lohoué–Rychener 1982]
- Damek–Ricci spaces [A–Damek–Yacoub 1996]

Heat kernel on $\mathbb{H}^n$

Sharp global estimate [Davies–Mandouvalos 1988]

$$h_t(r) \asymp t^{-\frac{n}{2}} (1+r) (1+t+r)^{\frac{n-3}{2}} e^{-(\frac{n-1}{2})^2 t} e^{-\frac{n-1}{2} r} e^{-\frac{r^2}{4t}}$$

$\forall\, t>0$ and $\forall\, r\geq 0$

Comments :

- spectral gap $\rightsquigarrow$ $e^{-(\frac{n-1}{2})^2 t}$
- Gaussian $e^{-\frac{r^2}{4t}}$
- $t\leq 1+r \rightsquigarrow t^{-\frac{n}{2}} j(r)^{-\frac{1}{2}}$
  where $j(r) = \left(\frac{\sinh r}{r}\right)^{n-1}$ jacobian of exponential map
- $t\geq 1+r \rightsquigarrow t^{-\frac{3}{2}} \phi_0(x)$
  where $\phi_0(r) \asymp (1+r)\, e^{-\frac{n-1}{2} r}$ fundamental spherical function

Heat kernel on G/K

- $h_t(x)$ bi–K–invariant Schwartz function on G
- no explicit expression in general
- tool $=$ inverse spherical Fourier transform :

$$h_t(x) = \text{const.} \int_{\mathfrak{a}} d\lambda \underbrace{|\mathbf{c}(\lambda)|^{-2}}_{\text{Plancherel measure}} e^{-t(|\rho|^2+|\lambda|^2)} \underbrace{\phi_\lambda(x)}_{\text{spherical functions}}$$

- Cartan decomposition $G = K(\exp \overline{\mathfrak{a}^+})K$
  $\rightsquigarrow$ estimate h_t on the positive Weyl chamber $\overline{\mathfrak{a}^+}$

Sharp global estimate [A–Ji 1999 ; A–Ostellari 2003]

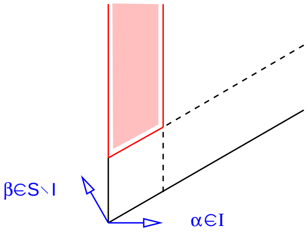
$$h_t(\exp X) \asymp t^{-\frac{n}{2}} \Big\{ \prod_{\alpha \in \Sigma^+_{\text{red}}} (1+\langle\alpha, X\rangle)\,(1+t+\langle\alpha, X\rangle)^{\frac{m_\alpha+m_{2\alpha}}{2}-1} \Big\} \times e^{-|\rho|^2 t}\, e^{-\langle\rho, X\rangle}\, e^{-\frac{|X|^2}{4t}}$$

$\forall\, t>0$ and $\forall\, X \in \overline{\mathfrak{a}^+}$

Heat kernel on G/K

- $h_t(\exp X) \asymp t^{-\frac{d_I}{2}}\, e^{-|\rho|^2 t}\, \phi_{I,0}(\exp X) \left(\frac{j_I(X)}{j(X)}\right)^{\frac{1}{2}} e^{-\frac{|X|^2}{4\,t}}$

  if t is bounded below and $\begin{cases} \langle \alpha, X\rangle \lesssim t & \forall\, \alpha \in I \\ \langle \alpha, X\rangle \gtrsim t & \forall\, \alpha \in S \smallsetminus I \end{cases}$



where $d_I = \operatorname{rank} G/K + 2\,|\Sigma^+_{I,\text{red}}| + \sum_{\alpha\in\Sigma^+\smallsetminus\Sigma^+_I} \dim \mathfrak{g}_\alpha$

Heat kernel on G/K

- $h_t(\exp X) \asymp t^{-\frac{d_I}{2}}\, e^{-|\rho|^2 t}\, \phi_{I,0}(\exp X)\, \left(\frac{j_I(X)}{j(X)}\right)^{\frac{1}{2}}\, e^{-\frac{|X|^2}{4\,t}}$

  if t is bounded below and $\begin{cases} \langle\alpha, X\rangle \lesssim t & \forall\, \alpha \in I \\ \langle\alpha, X\rangle \gtrsim t & \forall\, \alpha \in S \smallsetminus I \end{cases}$

- $h_t(x) \asymp t^{-\frac{d}{2}}\, e^{-|\rho|^2 t}\, \phi_0(x)\, e^{-\frac{|x|^2}{4\,t}}$

  if $t \gtrsim 1 + |x|$

- $h_t(\exp X) \asymp t^{-\frac{n}{2}}\, e^{-|\rho|^2 t}\, j(X)^{-\frac{1}{2}}\, e^{-\frac{|X|^2}{4\,t}}$

  if $t \lesssim 1 + \langle\alpha, X\rangle$ for all simple $\rightsquigarrow$ positive roots α

  for instance if t is bounded above

where $d = \operatorname{rank} G/K + 2\,|\Sigma^+_{\text{red}}|$ and $n = \dim G/K$

Green function on G/K

$$\begin{aligned} g_\zeta(x) &= \text{integral kernel of } (-\Delta - |\rho|^2 + \zeta^2)^{-1} \\ &= \int_0^{+\infty} ds\ e^{(\rho^2-\zeta^2)\,t}\, h_t(x) \end{aligned}$$

Behavior at infinity [A–Ji 1999]

$$g_\zeta(x) \asymp \begin{cases} |x|^{\frac{1-d}{2}}\,\phi_0(x)\,e^{-\zeta|x|} & \text{if } \zeta>0 \\ |x|^{2-d}\,\phi_0(x) & \text{if } \zeta=0 \end{cases}$$

Consequence [Guivarc'h–Ji–Taylor 1998] :
Martin compactification of G/K

Heat kernel on G

[Ostellari 2003]

$\Delta = \mathfrak{p}$–component of Casimir $+$ anything in $\mathfrak{k}$
Same estimate as for G/K when $t \geq 1+|x|$

[Alexopoulos–Lohoué 2003]

$\Delta = \sum_j X_j^2$ any invariant subLaplacian on G

- Upper estimate for $t \geq 1$ and $|x| = \mathrm{O}(t)$:
$$h_t(x) \leq C\, t^{-\frac{d}{2}}\, e^{-\Lambda t}\, \phi_0(x)\, e^{-c\frac{|x|^2}{t}}$$
- Lower estimate for $t \geq 1$ and $|x| = \mathrm{O}(\sqrt{t})$:
$$h_t(x) \geq C\, t^{-\frac{d}{2}}\, e^{-\Lambda t}\, \phi_0(x)$$

[A–Grigor'yan] : Extend lower estimate to $|x| = \mathrm{O}(t)$:
$$h_t(x) \geq C\, t^{-\frac{d}{2}}\, e^{-\Lambda t}\, \phi_0(x)\, e^{-c\frac{|x|^2}{t}}$$

Further results

- [Cowling–Giulini–Meda 1993] $L^{q_1} \longrightarrow L^{q_2}$ estimates for the heat semigroup $e^{t\Delta}$ on G/K
- [Bruno Schapira 2008] Heat kernel associated to the Heckman–Opdam Laplacian
- [...; A–Bougerol–Jeulin 2002] Brownian motion on G/K
- [Lalley 1991] Sharp global estimate for random walks on homogeneous trees
- [Parkinson 2007] Local and central limit theorems on affine buildings
- [A–Schapira–Trojan] Sharp global estimate for a simple random walk on affine buildings of type $\tilde{A_\ell}$
- [Steger–Trojan] Asymptotic behavior of random walks on affine buildings

Schrödinger equation on symmetric spaces

Schrödinger equation

$$\begin{cases} i\,\partial_t\,u(t,x)+\Delta_x\,u(t,x)=g(t,x) \\ u(0,x)=f(x) \end{cases}$$

Solution

$$u(t,x)=\underbrace{e^{\,i\,t\,\Delta_x}f(x)}_{\text{homogeneous}}+\underbrace{\int_0^t ds\;e^{\,i\,(t-s)\,\Delta_x}\,g(s,x)}_{\text{inhomogeneous}}$$

Homogeneous solution :

$$u(t,x)=e^{\,i\,t\,\Delta_x}f(x)=f*\underbrace{s_t}_{\text{Schrödinger kernel}}(x)$$

Schrödinger kernel on $\mathbb{H}^n$

Explicit expression

$$s_t(r) = 2^{-\frac{n+1}{2}}\,\pi^{-\frac{n}{2}}\,\underbrace{(i\,t)^{-\frac{1}{2}}}_{e^{-i\,\mathrm{sign}(t)\,\frac{\pi}{4}}\,|t|^{-\frac{1}{2}}}\,e^{-i\,(\frac{n-1}{2})^2\,t}\,\Big(-\frac{1}{\sinh r}\,\frac{\partial}{\partial r}\Big)^{\frac{n-1}{2}}\,e^{\,\frac{i}{4}\,\frac{r^2}{t}}$$

Global estimate

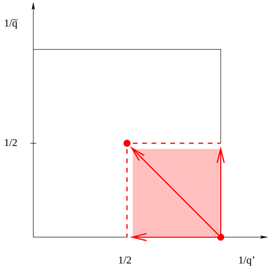
$$|s_t(r)| \lesssim \begin{cases} |t|^{-n/2}\,j(r)^{-1/2} & \text{if } |t|\le 1+r \\ |t|^{-3/2}\,\varphi_0(r) & \text{if } |t|\ge 1+r \end{cases}$$

2 fundamental inequalities

Dispersive estimate [A–Pierfelice]

Let $2 \leq q, \tilde{q} \leq \infty$

- $0 < |t| \leq 1 \,:\; \| e^{it\Delta} \|_{L^{q'} \to L^{\tilde{q}}} \lesssim |t|^{-\max\{\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{\tilde{q}}\}\, n}$
- $|t| \geq 1 \,:\; \| e^{it\Delta} \|_{L^{q'} \to L^{\tilde{q}}} \lesssim \begin{cases} 1 & \text{if } q = \tilde{q} = 2 \\ |t|^{-\frac{3}{2}} & \text{if } 2 < q, \tilde{q} \leq \infty \end{cases}$



2 fundamental inequalities

Hint : The crucial diagonal inequality

$$\| e^{it\Delta} \|_{L^{q'} \to L^q} \lesssim |t|^{-\frac{3}{2}} \quad \text{when} \quad \begin{cases} |t| \geq 1 \\ q > 2 \end{cases}$$

follows from the estimate

$$\int_0^{+\infty} dr \, |s_t(r)|^{\frac{q}{2}} \, \phi_0(r) \lesssim |t|^{-\frac{3}{2}}$$

and from the sharp Kunze–Stein phenomenon [Ionescu 2000]

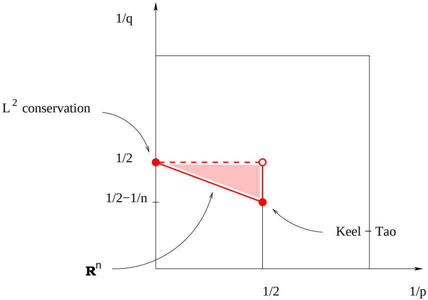
$$L^{2,1}(G) * L^{2,1}(G) \subset L^{2,\infty}(G)$$

2 fundamental inequalities

Strichartz estimate

$$\|u(t,x)\|_{L_t^{\tilde{p}}L_x^{\tilde{q}}} \lesssim \|f(x)\|_{L_x^2} + \|g(t,x)\|_{L_t^{p'}L_x^{q'}}$$

I $\forall\ (p,q),\ (\tilde{p},\tilde{q})$ in the *admissible* triangle



Application to NLS

Wellposedness $\sim$ existence & uniqueness

Consider the NLS

$$\begin{cases} i\,\partial_t\,u(t,x) + \Delta_x\,u(t,x) = \Gamma(u(t,x)) \\ u(0,x) = f(x) \end{cases}$$

with powerlike nonlinearities

$$\begin{cases} |\Gamma(u)| \lesssim |u|^{1+2\sigma} \\ |\Gamma(u)-\Gamma(v)| \lesssim |u-v|\,\{|u|^{2\sigma}+|v|^{2\sigma}\} \end{cases}$$

Then we have global wellposedness for small initial data

$$\begin{cases} \text{in } L^2(\mathbb{H}^n) & \text{if } \sigma \le \frac{2}{n} \\ \text{in } H^1(\mathbb{H}^n) & \text{if } \sigma \le \frac{2}{n-2} \end{cases}$$

Application to NLS

Particular case : $\Gamma(u) = u\,|u|^{2\sigma}$ defocusing
L^2 and H^1 conservation
LWP $\rightsquigarrow$ GWP for arbitrary large data and subcritical σ

Scattering

Same assumptions $\Rightarrow$ scattering for small initial data

$$\begin{cases} \text{in } \ L^2(\mathbb{H}^n) & \text{if } \ \sigma \le \frac{2}{n} \\ \text{in } \ H^1(\mathbb{H}^n) & \text{if } \ \sigma \le \frac{2}{n-2} \end{cases}$$

Comment :

Better dispersion in $\mathbb{H}^n$ $\rightsquigarrow$ stronger results than in $\mathbb{R}^n$

Further results

- Previous results under radial assumptions
  [Banica 2007 ; Pierfelice ; Banica–Carles–Staffinali]
- [Ionescu–Staffilani 2008] Related results in the defocusing case
  - weaker dispersive and Strichartz estimates
  - Morawetz inequality
  - scattering for arbitrary data
- [A–Pierfelice–Vallarino]
  - Damek–Ricci spaces (straightforward)
  - higher rank (in progress)
- [Kaizaku 2008]
  Smoothing effects on hyperbolic spaces
- [Burq–Guillarmou–Hassell]
  Convex cocompact surfaces with constant negative curvature

Wave equation on symmetric spaces

Wave equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) = g(t,x) \\ u(0,x) = f_0(x),\ \partial_t\big|_{t=0} u(t,x) = f_1(x) \end{cases}$$

Solution

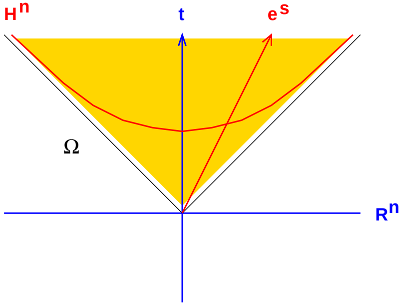
$$u(t,x) = \overbrace{\big(\cos t\sqrt{-\Delta_x}\big) f_0(x) + \frac{\sin t\sqrt{-\Delta_x}}{\sqrt{-\Delta_x}} f_1(x)}^{\text{homogeneous}}$$

$$+ \underbrace{\int_0^t ds\, \frac{\sin(t-s)\sqrt{-\Delta_x}}{\sqrt{-\Delta_x}}\, g(s,x)}_{\text{inhomogeneous}}$$

Tataru

- ◦ [Georgiev–Lindblat–Sogge 1997] GWP for NLW in $\mathbb{R}^n$
- • [Tataru 2001] shortcut via $\mathbb{H}^n$

Light cone Ω $\begin{cases} t^2 - x_1^2 - \cdots - x_n^2 > 0 \\ t > 0 \end{cases}$



Tataru

Wave operator (D'Alembertian)

$$\square = \partial_t^2 - \Delta_{\mathbb{R}^n} = e^{-\frac{n+3}{2}s} \circ \Big\{ \partial_s^2 \underbrace{- \Delta_{\mathbb{H}^n} - \big(\tfrac{n-1}{2}\big)^2}_{-\mathscr{L}_x} \Big\} \circ e^{\frac{n-1}{2}s}$$

Dispersive estimate [Tataru 2001]

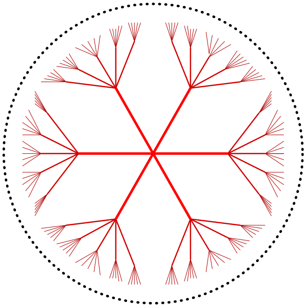
$$\Big\| (-\Delta)^{-\frac{n+1}{2}(\frac{1}{2}-\frac{1}{q})} \cos s\sqrt{-\mathscr{L}} \, \Big\|_{L^{q'} \to L^q} \lesssim |\sinh s|^{-(n-1)(\frac{1}{2}-\frac{1}{q})}$$

- Similar estimate for $\frac{\sin s\sqrt{-\mathscr{L}}}{\sqrt{-\mathscr{L}}}$
- Weighted Strichartz estimate on $\mathbb{H}^n$ $\rightsquigarrow$ on $\mathbb{R}^n$
- GWP for NLW on $\mathbb{R}^n$

Homogeneous trees

$X = \mathbb{T}_Q$ homogeneous tree with $Q+1$ edges

Example : $Q=5$



$\gamma = \frac{2}{Q^{1/2}+Q^{-1/2}}$

no local analysis $\rightsquigarrow$ no Sobolev spaces

Wave equation on $\mathbb{T}_Q$

$$\gamma\,\Delta_n^{\mathbb{Z}} u(n,x) - \underbrace{\{\Delta_x^X + 1 - \gamma\}}_{\mathscr{L}} u(n,x) = g(n,x)$$

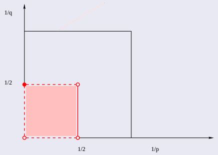
Laplacian on $\mathbb{Z}$: $\Delta^{\mathbb{Z}} f(n) = \frac{f(n+1)+f(n-1)-2f(n)}{2}$

Laplacian on X : $\Delta^X f(x) = \frac{1}{Q+1} \sum_{d(y,x)=1} f(y) - f(x)$

[A–Pierfelice–Vallarino]

Dispersive estimate : exponential decay in n

Strichartz estimate : valid for the full square



NLW : no critical power

Further results

- [Ionescu 2000]
  $L^q \to L^q$ estimates for $\cos t\sqrt{-\mathscr{L}}$ and $\frac{\sin t\sqrt{-\mathscr{L}}}{\sqrt{-\mathscr{L}}}$
  on hyperbolic spaces
- [Cowling–Giulini–Meda 2001 ; Cowling–Meda 2002]
  $L^{q_1} \to L^{q_2}$ estimates for $e^{-t\sqrt{-\Delta-|\rho|^2+\zeta^2}}$ on G/K
  for restricted complex time t
- [A–Pierfelice–Vallarino]
  $\mathbb{H}^n \begin{cases} \text{full dispersive and Strichartz estimates} \\ \text{sharp GWP for NLW} \end{cases}$